Lecture 10-25: Finite morphisms and normality

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We prove some basic facts about finite morphisms and normality that we will need to treat homogeneous spaces later.

First some basic definitions. Given a commutative ring A and an A-algebra B, recall that B is integral over A if every $b \in B$ satisfies a monic polynomial with coefficients in A. If B is finitely generated as an A-algebra, this condition is equivalent to requiring that B be finitely generated as an A-module. We say that B is of *finite type over A* in this situation. A morphism $\phi : X \to Y$ of affine varieties X, Y is called *finite* if the comorphism $\phi^* : \mathbf{k}[Y] \to \mathbf{k}[X]$ makes $\mathbf{k}[X]$ into a finitely generated $\mathbf{k}[Y]$ module. It is easy to show that any finite morphism is surjective.

More generally, we say that ϕ is *locally finite at* $x \in X$ if there is a finite morphism $\mu : Y' \to Y$ and an isomorphism ν of an open neighborhood U of x onto an open set in Y' such that $\mu \circ \nu$ is the restriction of ϕ to U. Now let $\psi : Y \to Z$ be another morphism of affine varieties.

Lemma 5.2.4, p. 83

If ϕ is locally finite at x and ψ is locally finite at $\phi(x)$ then the composite $\psi \circ \phi$ is locally finite at x.

Proof.

We may assume that $Y = D_{Z'}(f)$, where Z' is finite over Z and $f \in \mathbf{k}[Z']$. If Y' is finite over Y then $\mathbf{k}[Y'] = B_f$, where B is integral over $\mathbf{k}[Z']$. Hence B is integral over $\mathbf{k}[Z]$ and we have $Y' \cong D_V(g)$ for some V finite over Z and $g \in \mathbf{k}[V]$. □

Henceforth we assume that X, Y are irreducible and that ϕ is dominant, viewing $A = \mathbf{k}[Y]$ as a subring of $B = \mathbf{k}[X]$.

Lemma 5.2.5, p. 84

Assume that there is $b \in B$ with $B = A[b]$. Let $x \in X$. Then either $\phi^{-1}(\phi x)$ is finite and ϕ is locally finite at x or $\phi^{-1}(\phi x) \cong \mathbf{A}^1$.

Proof.

We have $B = A[T]/I$, where I is the ideal of polynomials $f \in A[T]$ with $F(b) = 0$. Let $\epsilon : A \to \mathbf{k}$ be the homomorphism defining the point ϕx. If $\epsilon I = 0$ then $\mathbf{k}[\phi^{-1}(\phi x)] = \mathbf{k}[T]$ and $\phi^{-1}(\phi x) \cong \mathbf{A}^1$. Otherwise the polynomials in ϵI vanish in $b(x)$, so that ϵI contains nonconstant polynomials and no nonzero constants. It follows that $\phi^{-1}(\phi x)$ is finite. It also follows that there is $f \in I$ of the form $f_n T^n + \ldots + f_m T^m + \ldots + f_0$, where $\epsilon(f_i) = 0$ if $i > m$ but $\epsilon(f_m) \neq 0$. Put $s = f_n b^{n-m} + \ldots + f_m$. Then $s \neq 0$ and $sb^m + f_{m-1} b^{m-1} + \ldots + f_0 = 0$, whence sb is integral over $A[s]$ and b is integral over $A[s^{-1}]$. Since $s \in A[b]$, s is integral over $A[s^{-1}]$ and thus also over A. Since $B_s = A[sb, s]_s$ the assertion follows. $\square$

Proposition 5.2.6, p. 84

Let $x \in X$. If the fiber $\phi^{-1}(\phi x)$ is finite then ϕ is locally finite at x; in particular, $\dim X = \dim Y$.

Proof.

We have $B = A[b_1, \ldots, b_h]$. If $h = 1$ the assertion holds by the previous result. Write $\phi = \phi' \circ \psi$, where $\psi : X \to X'$ and $\mathbf{k}[X'] = A[b_1]$. Clearly $\psi^{-1}(\psi x)$ is finite, whence by induction on h we may assume that ψ is locally finite at x. Then there is a morphism $\phi' : X" \to X'$ of affine varieties such that X is an affine open subset of $X"$ and ϕ is induced by ϕ'. Set $F = (\phi')^{-1}(\phi x)$. Assume that F is infinite; then $F \cong \mathbf{A}^1$. Let C be a component of $(\psi')^{-1}(F)$ of dimension at least 1 passing through x. Now $X \cap C$ is an open subset of C containing x, hence must be infinite. But $X \cap C$ lies in the finite set $\phi^{-1}(\phi x)$, a contradiction. Hence the components of $(\psi')^{-1}(F)$ of dimension at least 1 do not contain x. Replacing X by a suitable open neighborhood of x we may assume that no such components exist, so that F is finite. Then the result follows from the preceding one. □

There is a weaker notion than smoothness called *normality* which is crucial to the quotient construction in this chapter. It is (roughly) equivalent to smoothness in codimension one; that is, to the condition that the nonsmooth points of a variety form a subset of codimension at least two (though it is actually stronger than this condition).

Definition, p. 85

An integral domain A is called *normal*, or *integrally closed*, if every element of its quotient field integral over A already lies in A. A point x of an irreducible variety X is normal if there is an affine open neighborhood U of x such that $\mathbf{k}[U]$ is normal. We say that X is normal if all of its points are.

Note that a polynomial ring over a field, or more generally any unique factorization domain, is normal.

Theorem 5.28, p. 85: Zariski's main theorem

Let $\phi : X \to Y$ be a morphism of irreducible varieties that is bijective and birational and assume that Y is normal. Then ϕ is an isomorphism.

As an example, consider the morphism $x \mapsto x^p$, which is in fact a homomorphism either form G_a to itself for G_m to itself. This is bijective but not an isomorphism, as its inverse is not a morphism. Hence by Zariski this morphism cannot be birational, and indeed it is not, as the corresponding comorphism embeds one function field into a finite inseparable extension of itself.

Proof.

Let $x \in X$. Replace X, Y by affine open neighborhoods U, V of $x, \phi x$, respectively. Then U is isomorphic to an affine open subset of an affine variety V' that is finite over V. Now birationality implies that $\mathbf{k}(V') \cong \mathbf{k}(V)$, whence the normality of Y implies that the finite morphism $V' \to V$ is an isomorphism. Thus ϕ is an isomorphism of ringed spaces, hence an isomorphism of varieties. □

Lemma 5.2.10, p. 85

Let A be a normal integral domain with quotient field F. Let B be an integral domain that is an A-algebra of finite type. Assume that the quotient field E of B is separable over F. Then there is a nonzero element $a \in A$ such that the localization B_a is normal.

Proof.

It is well known that the integral closure $\overline{A}$ of A in E (set of all elements of E integral over A) is finitely generated as an A-module, since A is Noetherian (see e.g. Proposition 5.17, p. 64, in *Introduction to Commutative Algebra*, Atiyah-Macdonald). If $b_1, \ldots, b_n$ generate $\overline{A}$ over A, with the b_i nonzero, then one readily checks that $b = b_1 \ldots b_n$ has the desired property. □

Proposition 5.2.11, p. 86

Let X be an irreducible variety. The set of its normal points is nonempty and open.

It is clear that this set is open. Let $E = \mathbf{k}(X)$. We know by previous results that there is an affine open subset U of X with $\mathbf{k}[U]$ integral over a subalgebra A isomorphic to a poloynomial algebra and E separable over the quotient field of A. The previous result now shows that the set of normal points is nonempty.
This result also follows from the one proved before that X has a nonempty open subset of smooth points, since it is well know that any smooth point is normal.