Lecture 10-23: More about morphisms

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Following Chapter 5 in the text, we now study the behavior of dominant morphisms $\phi : X \to Y$ between irreducible varieties X, Y (so that the image ϕX is dense in Y, or equivalently the comorphism $\phi^* : \mathbf{k}[Y] \to \mathbf{k}[X]$ is injective). The goal throughout is to relate geometric properties of ϕ to algebraic ones of ϕ^*. We take our cue from the theory of vector spaces. It is well known that a linear map $\lambda : V \to W$ between a pair V, W of $\mathbf{k}$-vector spaces is such that the difference between the dimensions of V and the image of λ matches the dimension of the fiber of ϕ over every point in its image. The behavior of morphisms will be similar but not quite as nice. We will be spending most of our time in class on commutative algebra this week.

Begin with some purely field-theoretic definitions (p. 78). If F is a field and E is a finite extension of it we denote the degree of E over F by $[E : F]$. The elements of E separable over F form a subfield E_s, which is a separable extension of F; its degree $[E_s : F]$ is called the *separable degree* of E over F and is denoted $[E : F]_s$. Let p be the characteristic. If $p = 0$ we have $E = E_s$. If $p > 0$ then E is a purely inseparable extension of E_s, so that for all $x \in E$ there is an integer r with $x^{p^r} \in E_s$. Returning to morphisms, let $\phi : X \to Y$ be dominant. Recall from 1.8.1, p. 16, that the field of functions $\mathbf{k}(X)$ is well defined and coincides with $\mathbf{k}(U)$ for any open affine subset U of X; similarly $\mathbf{k}(Y)$ is defined. If $\dim X = \dim Y$, then $\mathbf{k}(X)$ is a finite extension of $\mathbf{k}(Y)$ (having transcendence degree 0). If $\mathbf{k}(X) = \mathbf{k}(Y)$ then ϕ is said to be *birational* and the varieties X, Y are *birationally equivalent*.

Lemma 5.1.2, p. 78

ϕ is birational if and only if there is a nonempty open subset U of X such that ϕU is open in Y and ϕ induces an isomorphism from U onto its image.

It is immediate that ϕ is birational if the condition of the lemma is satisfied. If conversely ϕ is birational, then we may assume that X and Y are affine. Then the coordinate ring $\mathbf{k}[X] = \mathbf{k}[Y][f_1, \ldots, f_r]$, where the f_i lie in $\mathbf{k}(Y)$. Take $f \neq 0$ in $\mathbf{k}[Y]$ such that $ff_i \in \mathbf{k}[Y]$ for all i. Then ϕ induces an isomorphism from the localization $\mathbf{k}[Y]_f$ to $\mathbf{k}[X]_f$ and the principal open set $U = D_X(f)$ has the required property.

The notion of birational equivalence has no counterpart in differential geometry. A simple example arises if we take X to be the affine line $\mathbf{A}^1$ and Y to be the twisted cubic in $\mathbf{A}^2$ defined by $x^3 = y^2$, with $\phi : X \to Y$ defined by $\phi(t) = (t^2, t^3)$. Here $\mathbf{k}(X) = \mathbf{k}(Y) = \mathbf{k}(t)$. Note that there is no morphism $\psi : Y \to X$ inverse to ϕ, though we do have what could be called a regular map from Y to X sending (x, y) to y/x if $x \neq 0$ and $(0, 0)$ to 0. Thus ϕ is birational and a homeomorphism but not an isomorphism, though it becomes an isomorphism when restricted to the open set $\mathbf{k}^*$.

We now prove three lemmas whose proofs will combine to give a proof of the main result. In each of them we assume that $\phi : X \to Y$ is a dominant morphism such that $\mathbf{k}[X] = \mathbf{k}[Y][f]$ is generated by $\mathbf{k}[Y]$ and a single polynomial f.

Lemma 5.1.3, p. 78

Suppose that f is transcendental over $\mathbf{k}(Y)$. Then ϕ is an open morphism (taking open sets to open sets) and if Y' is an irreducible closed subvariety of Y then $\phi^{-1}Y'$ is an irreducible closed subvariety of X of dimension $\dim Y' + 1$.

Proof.

We may assume that $X = Y \times \mathbb{A}^1$ and that ϕ is projection onto the first factor. Let $g = \sum_{i=0}^r g_i T^i \in \mathbf{k}[X] = \mathbf{k}[Y][T]$. Then $\phi(D_X(g)) = \cup_{i=0}^r D_Y(g_i)$, whence ϕ takes principal open subsets, and thus arbitrary open subsets, to open subsets. If Q is the prime ideal in $\mathbf{k}[Y]$ defined by the irreducible closed subvariety Y' then $\phi^{-1} Y'$ is the set of points of X at which the functions of the ideal $P = Q\mathbf{k}[X]$ vanish. Then $\mathbf{k}[X]/P \cong (\mathbf{k}[Y]/Q)[T]$. Since this last ring is an integral domain, P is prime and $\phi^{-1} Y'$ is irreducible. The last assertion is now clear. □

Lemma 5.1.4, p. 79

Assume that f is separably algebraic over $\mathbf{k}[Y]$. There is a nonempty open subset U of X with the following properties:

- The restriction of ϕ to U is open.
- If Y' is an irreducible closed subvariety of Y and X' is an irreducible component of $\phi^{-1}Y'$ that intersects U, then $\dim X' = \dim Y'$.
- For $x \in U$ the fiber $\phi^{-1}(\phi x)$ is a finite set with $[\mathbf{k}(X) : \mathbf{k}(Y)]$ elements.

Proof.

Now we have $\mathbf{k}[X] = \mathbf{k}[Y][T]/I$, where I is the ideal of polynomials vanishing at f. Let F be the minimal polynomial of f over $\mathbf{k}(Y)$. Choose $a \in \mathbf{k}[Y]$ such that all coefficients of F lie in $\mathbf{k}[Y]_a$. Let $f_1, \ldots, f_n$ be the roots of F in some extension field of $\mathbf{k}(Y)$. Since f is separable over $\mathbf{k}(Y)$ these roots are distinct and the discriminant $d = \Pi_{i<j}(f_i - f_j)^2$ is a nonzero element of $\mathbf{k}(Y)$ which can be expressed polynomially in the coefficients of F. It follows that there is $b \in \mathbf{k}[Y]$ and $m \geq 0$ with $a^m d = b$. We may replace X, Y, respectively, by $D_X(ab), D_Y(ab)$. □

Proof.

(continued) We are reduced to proving the lemma in the special case where I contains the minimum polynomial F. From this it follows, using the division algorithm, that I is exactly the ideal generated by F and that $\mathbf{k}[X]$ is a free $\mathbf{k}[Y]$-module. We can moreover assume that if $F(T) = \sum_{i=0}^{n} h_i T^i$ then for all $y \in Y$ the polynomial $F(y)(T) = \sum_{i=0}^{n} h_i(y) T^i$ has distinct roots. We will show now that the lemma holds with $U = X$. We may assume that $X = \{(y, t) \in Y \times \mathbb{A}^1 : F(y)(t) = 0\}$ and ϕ is the first projection map. Let $G \in \mathbf{k}[Y][T]$ and denote by g its image in $\mathbf{k}[X]$. Then $D_X(g) = \{(y, t) \in X : G(y)(t) \neq 0\}$. Write $G = QF + R$, where $R = \sum_{i=0}^{n-1} r_i T^i$ is a polynomial in T of smaller degree than n. Then $\phi D_X(g)$ is the set of all $y \in Y$ such that not all roots of $F(y)(T)$ are roots of $R(y)(T)$. Since the first polynomial has n distinct roots, this implies that $\phi D_X g = \cup_{i=0}^{n-1} D_Y(r_i)$ and the openness of ϕ follows. $\square$

Proof.

(continued) Next let Y' be as in the second assertion and let Q be the corresponding prime ideal in $\mathbf{k}[Y]$. Then $\phi^{-1}Y'$ is the closed set defined by the ideal $Q\mathbf{k}[X]$. Let $A = \mathbf{k}[Y]/Q$; denote by $\tilde{F}$ the image of F in $A[T]$. We claim that $Q\mathbf{k}[X]$ is a radical ideal. Let $H \in A[T]$ and assume that H^m is divisible by $\tilde{F}$. We may assume that the degree of H is less than n. We know that $\tilde{F}$ has distinct roots and that H is divisible by $\tilde{F}$, as polynomials in the quotient field of A. But since H has lower degree than $\tilde{F}$, this can only be if $H = 0$, proving the claim. □

Proof.

(continued) By primary decomposition, we know that $Q\mathbf{k}[X]$ is an intersection of prime ideals of $\mathbf{k}[X]$, say $\cap_{i=1}^r P_i$. We may assume that there are no inclusions among the P_i. The irreducible components of $\phi^{-1}Y'$ are the varieties $\mathcal{V}_X(P_i)$ corresponding to the P_i. We next show that $P_i \cap \mathbf{k}[Y] = Q$ for $1 \leq i \leq r$. If this fails, then say $P_1 \cap \mathbf{k}[Y] \neq Q$. Choose $x_1 \in P_1 \cap \mathbf{k}[Y] - Q, x_i \in P_1 - P_i$ for $2 \leq i \leq r$. Then $x_1 x_2 \ldots x_r \in Q\mathbf{k}[X]$. Since $\mathbf{k}[X]$ is free over $\mathbf{k}[Y]$, it follows that $x_2 \ldots x_r \in Q\mathbf{k}[X] \subset P_1$, which is impossible if $r > 1$, while if $r = 1$ we again get a contradiction, since $Q\mathbf{k}[X] \cap \mathbf{k}[Y] = Q$. It follows that the quotient field of $\mathbf{k}[X]/P_1$ is an algebraic extension of the quotient field of A, proving the second assertion. Finally, if Y' is a point then Q is a maximal ideal of $\mathbf{k}[Y]$ and $A = \mathbf{k}$. Then $\phi^{-1}Y'$ is the 0-dimensional variety with $\mathbf{k}$-algebra $\mathbf{k}[T]/(\tilde{F})$. Since $\tilde{F}$ has n distinct roots the third assertion follows. □

Lemma 5.1.5, p. 80

Let the characteristic p of $\mathbf{k}$ be positive and assume that $f^p \in \mathbf{k}(Y)$. There is a nonempty open subset U of X such that the restriction of ϕ to U is open and a homeomorphism onto its image. If Y' is an irreducible closed subvariety of Y then there is most one irreducible component X' of $\phi^{-1}Y'$ intersecting U. If X' exists it has the same dimension as Y'.

Let $f^p = g$. Replacing X, Y by $D_X(a), D_Y(a)$ for suitable $a \in \mathbf{k}[Y]$, we may assume that $g \in \mathbf{k}[Y]$ and that $\mathbf{k}[X]$ is free over $\mathbf{k}[Y]$. Then we can take $U = X$; the rest of the proof is similar to and easier than the previous one.

Finally we are ready for the payoff.

Theorem 5.1.6, p. 81

Let X, Y be irreducible varieties and $\phi : X \to Y$ a dominant morphism; set $r = \dim X - \dim Y$. Then there is a nonempty open subset U of X such that

- The restriction of ϕ to U is open.
- If Y' is an irreducible closed subvariety of Y and X' an irreducible component of $\phi^{-1}Y'$ intersecting U then $\dim X' = \dim Y' + r$; in particular, for $y \in Y$, any irreducible component of $\phi^{-1}(y)$ intersecting U has dimension r.
- if $\mathbf{k}(X)$ is algebraic over $\mathbf{k}(Y)$ then for all $x \in U$ the number of points in the fiber $\phi^{-1}(\phi x)$ is the separable degree $[\mathbf{k}(X) : \mathbf{k}(Y)]_s$.

Proof.

We can factor ϕ as the product of morphisms, each obeying the hypotheses of one of the preceding lemmas. These lemmas then yield the desired assertions. □

An easy argument yields the following stronger result.

Corollary 5.17, p. 82

In the situation of the previous result, for any variety Z the restriction of ϕ to U defines an open morphism from $U \times Z$ to $Y \times Z$.