Lecture 10-20: The Lie algebra and differentials

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Last time we gave the definition of the Lie algebra of an algebraic group G; now we discuss Lie algebras in more detail. We first show that the module Ω_G of differentials is completely controlled by the tangent space T_eG.

Proposition 4.4.2, p. 70

There is an isomorphism of $\mathbf{k}[G]$-modules $\Phi : \Omega_G \to \mathbf{k}[G] \otimes_{\mathbf{k}} (T_e G)^*$ such that $\Phi \circ \lambda(x) \circ \Phi^{-1} = \lambda(x) \otimes 1, \Phi \circ \rho(x) \circ \Phi^{-1} = \rho(x) \otimes (\mathrm{Ad}x)^*$. If $f \in \mathbf{k}[G]$ and $\Delta f = \sum_i f_i \otimes g_i$ then $\Phi(df) = -\sum_i f_i \otimes \delta g_i$. Here Δ is the comultiplication map $\mathbf{k}[G] \to \mathbf{k}[G] \otimes \mathbf{k}[G]$, so that $\Delta f(x, y) = f(xy)$, and $\delta f = f - f(e) + M_e^2$, as defined last time.

Proof.

The map sending (x, y) to (x, xy) is an automorphism of $G \times G$; the corresponding algebra automorphism Ψ of $A = \mathbf{k}[G]$ has $(\psi F)(x, y) = F(x, xy)$. Hence ψI is the ideal of functions vanishing on $G \times \{e\}$, which is $A \otimes M_e$, whence $\psi I^2 = A \otimes M_e^2$ and ψ induces a bijection of Ω_G onto $A \otimes (M_e/M_e^2)$. Let Φ be the composite of this bijection and the earlier isomorphism observed between T_eG and $(M_e/M_e^2)^*$. From the definition of ψ it follows that
$(\lambda(x) \otimes 1) \circ \psi = \psi \circ (\lambda(x), \lambda(x)), (\rho(x) \otimes c(x)) \circ \psi = \psi \circ (\rho(x) \otimes \rho(x))$, implying the first assertion. We also have
$\psi(f \otimes 1 - 1 \otimes f)(x, y) = \sum_i f_i(x)(g_x(e) - g_i(y))$, from which the second assertion follows. □

Recalling the notation $\mathcal{D} = \mathcal{D}_G$ for the $\mathbf{k}$-derivations of $A = \mathbf{k}[G]$ introduced last time, it follows at once that

Corollary 4.4.4, p. 71

There is an isomorphism $\Psi : \mathcal{D}_G \to \mathbf{k}[G] \otimes_{\mathbf{k}} T_e G$ of $\mathbf{k}[G]$-modules such that $\Psi \circ \lambda(x) \circ \Psi^{-1} = \lambda(x) \otimes 1, \Psi \circ \rho(x) \circ \Psi^{-1} = \rho(x) \otimes \mathrm{Ad}\, x$ for $x \in G$ and $\Psi^{-1}(1 \otimes X)(f) = -\sum_i f_i(Xg_i)$ for $X \in T_e G$.

Now we can apply these results to the Lie algebra $L(G)$. Let $\alpha_G = \alpha : \mathcal{D}_G \to T_eG$ be the map with $(\alpha_G D)(f) = (Df)(e)$.

Proposition 4.4.5, p. 71

α induces an isomorphism of vector space $L(G) \cong T_eG$ and if $x \in G$ we have $\alpha \circ \rho(x) \circ \alpha^{-1} = \mathrm{Ad}\, x$. In particular $\dim_{\mathbf{k}} L(G) = \dim G$.

Letting Ψ be as above we see that $\Psi(L(G) = 1 \otimes T_eG$ and $(\alpha \otimes \Psi^{-1})(1 \otimes X)(f) = -\sum_i f_i(e)(Xg_i) = -Xf$ since $f = \sum_i f_i(e)g_i$. The proposition follows at once.

A simple extension of this result shows that *for fixed $a \in G$ the the differential $d\psi_e$ at e of $\psi(x) = axa^{-1}x^{-1}$ is Ad a-1* (Lemma 4.4.13, p. 74)

Next let H be a closed subgroup of G. Denote by J the ideal of functions vanishing on H, so that $\mathbf{k}[H] = \mathbf{k}[G]/J$. Put $\mathcal{D}_{G,H} = \{D \in \mathcal{D}_g : DJ \subset J\}$. Then $\mathcal{D}_{G,H}$ is a Lie subalgebra of $\mathcal{D}_G$ and there is an obvious homomorphism of Lie algebras $\phi : \mathcal{D}_{G,H} \to \mathcal{D}_H$. We also have $T_eH = \{X \in T_eG : XJ = 0\}$. Then we get

Lemma 4.4.7, p. 72

ϕ defines an isomorphism of $\mathcal{D}_{G,H} \cap L(G)$ onto $L(H)$

.

Proof.

It follows from the definitions that $\alpha_H \circ \phi$ is the restriction of α_G to $\mathcal{D}_{G,H}$, whence ϕ is injective. To conclude it is enough to show that if $X \in T_e(H)$ then $D = \Psi^{-1}(1 \otimes X) \in \mathcal{D}_{G,H}$ with Ψ as above. if $f \in J$ and $\Delta f = \sum_i f_i \otimes g_i$ then we may assume for each i that one of the elements f_i or g_i lies in J. Then $Df \in J$, as required. □

Henceforth we identify the Lie algebra $L(G)$ with the tangent space T_eG, transferring the Lie algebra structure to the latter. We sometimes use German letters $\mathfrak{g}, \mathfrak{h}, \ldots$ for the Lie algebras of $G, H, \ldots$. Given a homomorphism $\phi : G \to G'$ of algebraic groups its differential $d\phi$ is easily seen to be a homomorphism of Lie algebras $L(G) \to L(G')$ (Proposition 4.4.9, p. 72).

Example

This is Example 4.4.10 on p. 73. First let $G = G_a, \mathbf{k}[G] = \mathbf{k}[T]$. The derivations of G commuting with translations $T \to T + a$ are just the multiples of $X = \frac{d}{dT}$. If $p > 0$ we have $X^p = 0$. So $\mathfrak{g}$ is the one-dimensional Lie algebra $\mathbf{k}X$ with trivial bracket and trivial pth power operation if $p > 0$. Next let $G = G_m$; here $\mathbf{k}[G] = \mathbf{k}[T, T^{-1}]$. Now the derivations commuting with the translations $T \to Ta$ are the multiples of $Y = T\frac{d}{dT}$. Once again $\mathfrak{g}$ is one-dimensional, with trivial bracket, but now $Y^p = Y$, so that the pth power operation is different, if $p > 0$.

Example

Continuing with Example 4.4.10, let $G = GL_n, \mathbf{k}[G] = \mathbf{k}[T_{ij}, D^{-1}]$ for $1 \leq i, j \leq n$, where D is the determinant. Here G is an open subset of $\mathfrak{gl}_n$, the set of $n \times n$ matrices, so that $\mathfrak{g} = \mathfrak{gl}_n$ as a vector space. If $X = (T_{ij}) \in \mathfrak{g}$, then $D_X T_{ij} = \sum_{h=1}^{n} T_{ih} x_{hj}$ defines a derivation of $\mathbf{k}[G]$ commuting with left translations, which thus lies in $\mathfrak{g}$. Hence $\mathfrak{g}$ consists exactly of the D_X; the bracket operation is given by commutation of matrices. The pth power operation sends a matrix to its pth power in the ordinary sense, if $p > 0$. We have $\mathrm{Ad}(x)X = xXx^{-1}$ for $x \in G, X \in \mathfrak{g}$. Finally, if H is a closed subgroup of G, then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

In the theory of Lie groups, one establishes a bijection between connected subgroups of a (real or complex) Lie group and subalgebras of its Lie algebra; one also shows that closed subgroups of a Lie group are Lie subgroups, but not conversely. Neither of these results holds for linear algebraic groups. First of all, we consider only closed subgroups of a given algebraic group, each one having a Lie algebra that is a subalgebra of the ambient Lie algebra, but *not* all Lie subalgebras arise in this way. (The ones that do are called *algebraic* and it is still not completely settled what the algebraic subalgebras of $\mathfrak{gl}(n)$ are.) In particular, there is no algebraic group analogue of the exponential map in manifold theory.

I conclude with some simple differentiation formulas. Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Let $\mu : G \times G \to G$ and $i : G \to G$ be the multiplication and inverse maps, respectively, on G.

Lemma 4.4.12, p. 74

We have $(d\mu)_{(e,e)}(X, Y) = X + Y, (di)_e(X) = -X$ for $X, Y \in \mathfrak{g}$.

The multiplication map μ defines a linear map $\tilde{\mu} : \Omega_G \to \Omega_{G\times G} = (\Omega_G \otimes \mathbf{k}[G]) \oplus (\mathbf{k}[G] \otimes \Omega_G)$. If $f \in \mathbf{k}[G], \Delta f = \sum_i f_i \otimes g_i$, then $\tilde{\mu}(df) = \sum(df_i \otimes g_i + f_i \otimes dg_i$. Since $f = \sum f_i(e)g_i = \sum g_i(e)f_i$ we have that $\tilde{\mu}(df) - df \otimes 1 - 1 \otimes df \in M_{e,e}\Omega_{G\times G}$. Hence the linear map of $\Omega_G(e)$ to $\Omega_{G\times G}(e, e,) = \omega_G(e) \oplus \Omega_G(e)$ induced by $\tilde{\mu}$ sends u to (u, u). As $(d\mu)_{e,e}$ is the dual of this map, the first assertion follows. The second follows from the fact that $\mu \circ (\mathrm{id}, i)$ is the trivial map sending G to $\{e\}$.

Finally, let G_1, G_2 be two linear algebraic groups acting linearly on the vector spaces V_1, V_2. Then the tensor product $V_1 \otimes_{\mathbf{k}} V_2$ carries a natural $G_1 \times G_2$ action, for which $(g_1, g_2).(v_1 \otimes v_2) = g_1.v_1 \otimes g_2.v_2$. The differentiated action on $L(G_1 \times G_2) = L(G_1 \oplus L(G_2)$ has $(X_1, X_2).(v \otimes w) = X_1 \cdot v \otimes w + v \otimes X_2.w$. If the V_i are finite-dimensional and each G_i acts irreducibly on V_i (so that no proper subspace of V_i is stable under G_i), then $V_1 \otimes V_2$ is irreducible under the $G_1 \times G_2$ action and every finite-dimensional irreducible representation of $G_1 \times G_2$ arises in this way.

Taking $G = G_1 = G_2$ and specializing to the diagonal subgroup of $G \times G$, we get for any rational representations V, W of G a representation of G on $V \otimes W$, but this time this representation is not necessarily irreducible even if V and W are. The differentiated action of the representation of $L(G)$ on $V \otimes W$ has $X.(v \otimes w) = X.v \otimes w + v \otimes X.w$. In particular, we get natural actions of G and $L(G)$ on the nth tensor power $T^n V = v^{\otimes n}$ as well as on its quotients the nth symmetric power $S^n V$ and the nth exterior power $\wedge^n V$.