Lecture 10-2: Prevarieties and varieties

October 2, 2023

Continuing from last time, let ϕ be a morphism from the affine variety $X \subset \mathbf{k}^n$ to $\mathbf{k}^m$. Set $Y = \overline{\phi X}$, an affine variety in $\mathbf{k}^m$. We have seen that the homomorphism $\phi^* : \mathbf{k}[Y] \to \mathbf{k}[X]$ is one-to-one. We need the following result from commutative algebra.

Proposition 1.9.4, p. 19

Let B be an integral domain and A a subring such that B is finitely generated as an A-algebra. Given $b \neq 0$ in B there is $a \neq 0$ in A such that any homomorphism π from A to an algebraically closed field F with $\pi(a) \neq 0$ can be extended to a homomorphism π from B into F with $\pi(b) \neq 0$.

I refer to the text for the proof. As you will have observed by now, I will generally omit proofs from commutative algebra, referring instead to Springer or some other source. I prefer to concentrate instead on the geometric consequences of algebraic results.

Theorem 1.9.5, p. 19

In the above setting, the image ϕX contains a nonempty open subset of $Y = \overline{\phi X}$.

Letting $X_1, \ldots, X_r$ be the irreducible components of X we see that it suffices to assume that X is irreducible. Choosing any nonzero $b \in \mathbf{k}[X]$ let $a \in \mathbf{k}[Y]$ satisfy the conclusion of the lemma. Then the principal open subset $D(a)$ of Y corresponding to a lies in the image of ϕ, since a maximal ideal of $\mathbf{k}[Y]$ not containing a corresponds to a point y of Y on which a does not vanish, which lies in a maximal ideal of $\mathbf{k}[X]$, which in turn corresponds to a point $x \in X$ with $\phi(x) = y$.

Note however that the image ϕX need not itself be open in its closure (nor need it be an affine variety). As an example, consider the morphism π from $\mathbf{k}^2$ to itself sending (x, y) to (x, xy). Here the image is the union of $\mathbf{k}^2$ with the y-axis removed and the origin.

Affine varieties belong to a larger class of topological spaces which are locally affine in the same way that manifolds are locally Euclidean.

Definition 1.6.1, p. 11

A prevariety X is a quasicompact ringed space $(X, \mathcal{O}_X)$ (or simply X) such that any point of X has an open neighborhood U such that the induced ringed space U is isomorphic to the ringed space of an affine variety. We call the sets U affine open.

This is a special case of an even more general kind of object called a scheme.

For example, any quasi-affine variety (open subset of an affine variety V) is a prevariety, since it is the union of principal open subsets, each of them affine, since the coordinate ring of each is obtained from the coordinate ring of V by localizing by the powers of a single polynomial. (On the other hand, quasi-affine varieties are not in general affine; for example, let X be the affine space $\mathbf{A}^2$ with the origin removed. The ring $\mathcal{O}_X(X)$ of regular functions on X is easily seen to coincide with that $\mathbf{A}^2$, whence the natural map from maximal ideals of this ring to points of X is not bijective.)

I now want to define the notion of variety; to do this I need the product construction. Before giving this construction, I make the elementary observation that the Cartesian product $X \times Y$ of affine varieties $X \subset \mathbf{k}^n, Y \subset \mathbf{k}^m$ is an affine subvariety of $\mathbf{k}^{n+m}$ in an obvious way: if the polynomials $f_1(x_1, \ldots, x_n), \ldots f_r(x_1, \ldots, x_n)$ generate the ideal of definition of X and $g_1(y_1, \ldots, y_m), \ldots, g_s(y_1, \ldots, y_m)$ generate the corresponding ideal for Y, then the f_i and g_j, regarded as functions of $x_1, \ldots, x_n, y_1, \ldots y_m$, generate the ideal of definition of $X \times Y$. The coordinate ring $\mathbf{k}[X \times Y]$ coincides with the tensor product $\mathbf{k}[X] \otimes_{\mathbf{k}} \mathbf{k}[Y]$. Note that the Zariski topology on $\mathbf{k}^{n+m} = \mathbf{k}^n \times \mathbf{k}^m$ is *not* the product of the Zariski topologies on $\mathbf{k}^n$ and $\mathbf{k}^m$.

Proposition 1.6.3, p. 11

Given two prevarieties X, Y, we make their Cartesian product $X \times Y$ into a prevariety by first writing X, Y as the respective finite unions of affine open sets $\cup_{i=1}^m U_i, \cup_{j=1}^n V_j$. Make each product $U_i \times V_j$ into an affine variety as above. We then declare a subset of $X \times Y$ to be open if and only if its intersection with each product $U_i \times V_j$ is open. A **k**-valued function f on an open neighborhood of $x \in U_i \times V_j$ is regular if and only if its restriction to $U_i \times V_j$ is regular as a function on an affine variety for all i, j.

It is easy to check that the axioms of a ringed space are satisfied by $X \times Y$ and that $X \times Y$ is irreducible (as a topological space) whenever X and Y are.

Definition 1.6.9, p. 12

A prevariety X is called a (separated) variety if the diagonal $\Delta_X = \{(x,x) : x \in X\}$ is a closed subset of the product $X \times X$.

It is easy to see that any affine or quasi-affine variety is a variety in this sense. Note that the image $\pi\mathbf{k}^2$ of $\mathbf{k}^2$ under the morphism π on $\mathbf{A}^2$ defined above is not even a prevariety, as the origin has no affine open neighborhood.

We have the following criterion for a prevariety to be a variety.

Proposition 1.6.12, p. 13

A prevariety X equal to the finite union $\cup_{i=1}^{m} U_i$ of affine open subsets U_i is a variety if and only if for each pair (i, j) the intersection $U_i \cap U_j$ is affine open (or empty) and the images under the restriction maps of $\mathcal{O}_X(U_i)$ and $\mathcal{O}_X(U_j)$ in $\mathcal{O}_X(U_i \cap U_j)$ generate this algebra.

Example

Using this criterion I can give the simplest example of a prevariety that is not a variety (see Exercise 1.6.13, p. 13). Let the set X be the union of $\mathbf{A}^1$ and a point $\{0'\}$. Give $\mathbf{A}^1 \subset X$ the obvious structure of affine variety and declare to be affine open in X. Define $\phi : \mathbf{A}^1 \to X$ via $\phi(x) = x$ if $x \in \mathbf{A}^1, x \neq 0$, while $\phi(0) = 0'$. Declare $\phi\mathbf{A}^1$ to be affine open with the transported structure of variety, so that $\{\mathbf{A}^1, \phi\mathbf{A}^1\}$ is an affine open cover of X. Then X is not a variety, since the intersection $\mathbf{A}^1 \cap \phi\mathbf{A}^1 = \mathbf{A}^1 \setminus \{0\}$ has as its ring of regular functions the localization $\mathbf{k}[x]_x$ at x, while the restrictions of the coordinate rings of both $\mathbf{A}^1$ and $\phi\mathbf{A}^1$ generate only $\mathbf{k}[x]$.

Example

On the other hand, a closely related example gives rise to a non-affine variety (continuing with Exercise 1.6.3 on p. 13). Take $\mathbf{P}^1$ to be the union $\mathbf{A}^1 \cup \{\infty\}$. Define ϕ on $\mathbf{A}_1 \subset \mathbf{P}^1$ via $\phi(x) = x^{-1}$ for $x \neq 0, \phi(0) = \infty$. This time the criterion to be a variety is satisfied, since the restrictions of the relevant rings of regular functions rings generate the regular functions on the corresponding intersection of open subsets. Note also that the only regular functions defined on all of $\mathbf{P}^1$ are constants.

I conclude by remarking that the above theorem (that the image of a morphism of affine varieties contains a nonempty open subset of its closure) extends to morphisms of prevarieties, as one sees by covering them by affine open subsets.