# Lecture 10-18: Existence of smooth points and definition of the Lie algebra

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Now we can show that while a variety can have singular points, most points on it are smooth; for algebraic groups every point is smooth. We then globalize the definition of tangent space to define the Lie algebra of an algebraic group.

Retain the notation of last time: $F, E'$, and $E$ are fields with $F \subset E' \subset E$ and $\Omega_{E/E'}, \Omega_{E/F}$ are the respective modules of differentials of $E$ over $E'$ and $F$. We say that $E$ is *separably generated* over $F$ (p. 63) if there is a purely transcendental extension $E'$ of $F$ contained in $E$ such that $E$ is separably algebraic over $E'$; this condition is automatic unless the characteristic $p$ of $\mathbf{k}$ is positive. Write $t = \text{trdeg}_F E$, the transcendence degree of $E$ over $F$.

**Theorem 4.2.9, p. 63**

We have $\dim_E \Omega_{E/F} \geq t$ with equality if and only if $E$ is separably generated over $F$.

## Proof.

Let $E$ be generated as a field by $x_1, \ldots, x_m$ over $F$. We argue by induction on $d = \dim_E \Omega_{E/F}$. If $d = 0$ and $m = 1$ then we proved this result last time. If $m > 1$ then the exact sequence with $\alpha$ proved last time, with $E' = F(x_1)$, shows that $\Omega_{E/F(x_1)} = 0$. By induction on $m$ we may assume that $E$ is separably algebraic over $F(x_1)$. Injectivity of $\alpha$ forces $\Omega_{F(x_1)/F} = 0$, whence $x_1$ is separable over $F$ and $E$ is separably algebraic over $F$ and the result holds for $d = 0$. By induction on $m$ one also shows that $\Omega_{E/F} = 0$ if $E$ is separably algebraic over $F$. □

## Proof.

Now let $d > 0$ and assume the result holds for smaller values. We know that there is $x \in E$ with $d_{E/F}x \neq 0$. Apply the exact sequence again with $E' = F(x)$. Since $\alpha(1 \otimes d_{F(x)/F}x) = d_{E/F}x \neq 0$ we have $\Omega_{F(x)/F} \neq 0$, whence $\dim_{F(x)} \Omega_{F(x)/F} = 1$ and $\alpha$ is injective. Hence $\dim_E \Omega_{E/F} = \dim_E \Omega_{E/F(x)} + 1$. By inductive hypothesis we have $\dim_E \Omega_{E/F} \geq \text{trdeg}_{F(x)} E + 1$. By the additivity of transcendence degree and $\text{trdeg} F(x) \leq 1$, the first assertion follows. If equality holds then $x$ is transcendental over $F$ and by induction $E$ is separably generated over $F(x)$, hence also over $F$. □

**Proof.**

It remains to show that if $E$ is separably generated over $F$ then equality holds. Apply the exact sequence for $E'$ to a purely transcendental extension over which $E$ is separably algebraic. We know that $\Omega_{E/E'} = 0$. Then $\dim_E \Omega_{E/F} = \dim_{E'} \Omega_{E'/F} = \mathrm{trdeg}_F E' = \mathrm{trdeg}_F E$, as desired. □

We call $E$ separable over $F$ if either $p = 0$ or $p > 0$ and any elements $x_1, \ldots, x_s$ of $E$ linearly independent over $F$ are such that $x_1^p, \ldots, x_s^p$ are also linearly independent over $F$. If $F$ is *perfect*, so that either $p = 0$ or every element of $F$ is a $p$th power, then all extensions of $F$ are separable; in particular, this holds if $F$ is algebraically closed.

A similar argument to that of the previous theorem shows that *any separable extension is also separably generated* (Proposition 4.2.10, p. 64), whence *if F is perfect then an extension E of* $E' \supset F$ *is separably generated if and only if* $\alpha : E \otimes_{E'} \Omega_{E'/F} \to \Omega_{E/F}$ *is injective, or if and only if the map* $Der_F(E, E) \to Der_F(E', E)$ *is surjective* (Corollary 4.2.11, p. 64).

The main result on tangent spaces is

**Theorem 4.3.3, p. 67**

Let $X$ be an irreducible variety of dimension $e$.

- If $x$ is a simple point of $X$ then there is an affine open neighborhood $U$ of $x$ such that $\Omega_U = \Omega_{\mathbf{k}[U]/\mathbf{k}}$ is a free $\mathbf{k}[U]$-module with basis $dg_1, \ldots, dg_e$ for some $g_i \in \mathbf{k}[U]$.
- The simple points of $X$ form a nonempty open subset of $X$.
- For any $x \in X$ we have $\dim_{\mathbf{k}} T_x X \geq e$.

## Proof.

We may assume that $X$ is affine and that $\mathbf{k}[X] = \mathbf{k}[T_1, \ldots, T_m]/(f_1, \ldots, f_n)$. Let $J$ be the Jacobian matrix of the $f_i$. Regarding its elements as lying in the function field $\mathbf{k}(X)$, let $r$ be the largest integer such that some $r \times r$ submatrix of $J$ does *not* have an identically vanishing determinant. Then the set of $x \in \mathbf{k}^m$ for which the corresponding $r \times r$ submatrix of $J(x)$ has nonzero determinant is open and the intersection $U'$ of this set and $X$ is open in $X$. Then clearly the dimension of the tangent space of $X$ at any point of $U'$ is $m - r$, while the dimension of this space at any other point of $X$ is larger. A simple argument using elementary row and column operations (see 4.2.15, p. 66) shows that there is a principal open set $D(f)$ such that the intersection $U$ of $U'$ and this set satisfies the first assertion, as desired. □

There are two important consequences of this theorem, one for morphisms of irreducible varieties and the other for $G$-spaces with $G$ an algebraic group. Recall first that a morphism $\phi : X \to Y$ with the image $\phi X$ dense in $Y$ is such that the coordinate ring $\mathbf{k}[Y]$ embeds in $\mathbf{k}[X]$, whence the function field $\mathbf{k}(Y)$ likewise embeds in $\mathbf{k}(X)$. We say that $\phi$ is *dominant* in this case; we also say that $\phi$ is *separable* if the extension $\mathbf{k}(X)$ is separably generated over $\mathbf{k}(Y)$.

## Theorem 4.3.6, p. 68

Let $\phi : X \to Y$ be a morphism of irreducible varieties. If $x$ is a simple point of $X$ such that $\phi x$ is a simple point of $Y$ and $d\phi_x$ is surjective, then $\phi$ is dominant and separable. If $\phi$ is dominant and separable then the set of simple points of $X$ with the properties of the previous assertion form a nonempty open subset of $X$.

## Proof.

By the previous result we may replace $X, Y$ by suitable affine open subsets so that both are smooth; then $\Omega_X, \Omega_Y$ are free modules over $\mathbf{k}[X], \mathbf{k}[Y]$, respectively, of ranks $d = \dim X, e = \dim Y$. We then get a homomorphism of free $\mathbf{k}[X]$-modules $\psi : \mathbf{k}[X] \otimes_{\mathbf{k}(Y)} \Omega_Y \to \Omega_X$ (see 4.3.5, p. 68). Fixing bases of these modules, $\psi$ is described by a $d \times e$ matrix $A$ with entries in $\mathbf{k}[X]$. Now let $x \in X$ be such that $d\phi_X$ is surjective. Then the matrix $A(x)$ has rank $e$, whence the rank of $A$, as a matrix over $\mathbf{k}(X)$ is at least $e$. Since the rank is at most $e$, it must be exactly $e$, whence $e$ is injective. Then the homomorphism $\phi^* : \mathbf{k}[Y] \to \mathbf{k}[X]$ is injective (since $\Omega_X, \Omega_Y$ are free), so that $\phi$ is dominant. The homomorphism $\alpha$ from last time with $E = \mathbf{k}(X), E' = \mathbf{k}(Y)$ is also injective (having the matrix $A$), whence $\phi$ is separable. Conversely, if $\phi$ is dominant and separable, then the rank of $A$ over $\mathbf{k}(X)$ is $e$, whence the set of $x \in X$ with the rank of $A(x)$ equal to $e$ is open and nonempty. □

Unlike the situation for smooth manifolds it is possible that $d\phi$ is surjective on an open subset but not on all of $X$; an example is given by the morphism $\mathbf{k} \to \{(x, y) \in \mathbf{k}^2 : x^3 = y^2\}$ sending $t$ to $(t^2, t^3)$, whose differential fails to be surjective at $t = 0$ (and the image of this point is not smooth). For connected groups acing on varieties, however, the story is more uniform. This is

## Theorem 4.3.7, p. 69

Let $G$ be a connected algebraic group.

- Any homogeneous space $X$ for $G$ is irreducible and smooth; in particular, $G$ is smooth
- Let $\phi : X \to Y$ be a $G$-morphism of homogeneous spaces (commuting with the action of $G$). Then $\phi$ is separable if and only if $d\phi_x$ is surjective for one $x \in X$, or if and only $d\phi_x$ is surjective for all $x \in X$.
- Let $\phi : G \to G'$ be a surjective homomorphism of algebraic groups. Then $\phi$ is separable if and only if $d\phi_e$ is surjective (where $e$ is the identity).

This follows at once from the previous result and remarks made last time about tangent spaces of $G$-varieties.

Again let $G$ be an algebraic group. Denote by $\lambda, \rho$ the operations of left and right translations in $A = \mathbf{k}[G]$. View $A \otimes A$ as the algebra $\mathbf{k}[G \times G]$. If $m : A \otimes A \to A$ is the multiplication map then for $F \in \mathbf{k}[G \times G]$ we have $(mF)(x) = F(x, x)$. So $I = \ker m$ is the ideal of functions on $G \times G$ vanishing on the diagonal. For $x \in G$ the automorphisms $\lambda(x) \otimes \lambda(x)$ and $\rho(x) \otimes \rho(x)$ of $\mathbf{k}[G \times G]$ stabilize $I$ and $I^2$, so induce automorphisms of $\Omega_G = I/I^2$, also denoted by $\lambda(x), \rho(x)$. We thus get representations $\lambda, \rho$ of $G$ in $\Omega_G$ which are locally finite. For $x \in G$ the map $c(x) : y \to xyx^{-1}$ is an automorphism of $G$ fixing $e$. It induces linear automorphisms of the tangent space $T_eG$ and its dual space $(T_eG)^*$, denoted by Ad $x$ and $(\text{Ad } x)^*$, respectively. For $u \in (T_eG)^*$ we have $((\text{Ad } x)^* u)X = u(\text{Ad}(x^{-1}X)$ for $x \in G, X \in T_eG$. See 4.4.1, p. 69. The action of $G$ on $T_eG$ is called the *adjoint representation*.

Let $M_e \subset A$ be the maximal ideal of functions vanishing at $e$. We have seen that the dual space $(T_eG)^*$ can be identified with $M_e/M_e^2$. If $f \in A$ we denote by $\delta f$ the element $f - f(e) + M_e^2$ of $(T_eG)^*$. For $X \in T_eG =$Der$_k(A, \mathbf{k}_e)$ we have $(\delta f)(X) = Xf$.

## Definition 4.4.3, p. 71

Denoting by $\mathcal{D}$ the algebra $\mathrm{Der}_{\mathbf{k}}(\mathbf{k}[G], \mathbf{k}[G])$ we denote by $L(G)$ the space of derivations commuting with all $\lambda(x)$ for $x \in G$ and call it the *Lie algebra* of $G$.

The Lie algebra $\mathfrak{g} = L(G)$ is the honest tangent space to $G$ promised in the last lecture. The commutator, or (Lie) bracket, $[D, D'] = DD' - D'D$ of two derivations in $L(G)$ is easily seen to lie in $L(G)$; if the characteristic $p$ of $\mathbf{k}$ is positive, then the $p$th power $D^p$ of a derivation in $L(G)$ also lies in $L(G)$. Hence $L(G)$ is indeed a Lie algebra over $\mathbf{k}$ by the definition that one sees in manifold theory. In fact, in characteristic $p > 0$ it is what is called a *restricted* Lie algebra, meaning that it has a $p$th power map satisfying something called Jacobson's formula (see p. 71).