Lecture 10-16: Tangent spaces: heuristic motivation, formal definition, and properties

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Like differentiable manifolds, algebraic varieties have tangent spaces at every point; but unlike manifolds, varieties can have singular points at which the tangent space has dimension larger than that of the variety. Following Chapter 4 in the text, I will give the formal definition of tangent spaces and develop some of their properties

I begin with some motivation. Let $X \subset \mathbf{k}^n$ be an affine variety whose ideal I of definition is generated by $f_1, \ldots, f_s$. Let $x \in X$ and let $L = \{x + tv : t \in \mathbf{k}\}$ be a line through x in $\mathbf{k}^n$, where $v = (v_1, \ldots, v_n)$ is a direction vector for L. Writing out the Taylor series of each $f_i(x + tv)$ we get $f_i(x + tv) = t \sum_{j=1}^n v_j (D_j f_i)(x) + t^2(\ldots)$, where D_j denotes partial differentiation with respect to the jth variable x_j, whence $t = 0$ is a multiple root of the equations $f_i(x + tv) = 0$ if and only if $\sum_{j=1}^n v_j (D_j f_i)(x) = 0$. If so we call L a tangent line and v a tangent vector to X at x. The tangent space of X at x is the set of all tangent vectors of X at x. This definition will be reformulated slightly when we make it more formal; the tangent space at a point will be a quotient rather than a subspace of a $\mathbf{k}$-vector space. We will also attach a kind of tangent object to all of X, though it will be too large to regard as a tangent space to it. If $X = G$ is an algebraic group we will shrink this tangent object to a genuine tangent space to X.

Note that in the special case $n = 3$, taking $\mathbf{k} = \mathbb{R}$, and replacing $f_1, \ldots, f_s$ by a single differentiable function f of three variables, this is just the Math 126 definition of the tangent space to the level surface defined by $f = 0$ in $\mathbb{R}^3$; but note that a key difference from the situation in Math 126 is that if $\nabla f(x) = 0$ then rather than saying that the tangent space of $f = 0$ is undefined at x we now say that it is all of $\mathbb{R}^3$ there. The operator $D' = \sum v_j D_j$ is then a *derivation* from the polynomial algebra $T = \mathbf{k}[x_1, \ldots, x_n]$ into itself; that is, it is a $\mathbf{k}$-linear map such that $D(fg) = (Df)g + (D(g)f$. The requirement above on the v_j that $\sum_{j=1}^{n} v_j D_j(f_i)(x) = 0$ for every generator f_i of I translates to the requirement that $DI \subset M_x$, where $M_x \subset \mathbf{k}[X]$ is the maximal ideal of functions vanishing at x.

Evaluating all functions at x, we see that D can be regarded as a function from $\mathbf{k}[X]$ to k (rather than to $\mathbf{k}[X]$) such that $D(fg) = (Df)g(x) + (Dg)f(x)$. We all such a map a $\mathbf{k}$*-derivation of* $\mathbf{k}[X]$ *in* $\mathbf{k}_x$; here the range $\mathbf{k}_x = \mathbf{k}$ is regarded as a $\mathbf{k}[X]$-module via $f.k = f(x)k$ for $f \in \mathbf{k}[X], k \in \mathbf{k}_x$. Conversely, any $\mathbf{k}$-derivation of $\mathbf{k}[X]$ in $\mathbf{k}_x$ arises in this way, since any such map D sends all constants to 0, is completely determined by the images $v_i = Dx_i$ of the variables x_i, and the requirement on the v_i is exactly that $\sum_{j=1}^{n} v_j(D_j f_i)(x) = 0$ for all generators f_i of I (or equivalently for all elements of I). Clearly this definition does not depend on the choice of the generators f_i. We thus define T_xX to be the set $\mathrm{Der}_{\mathbf{k}}(\mathbf{k}[X], \mathbf{k}_x)$ of such derivations. (p. 58)

In general (4.1.1, p. 57), given a commutative ring R, an R-algebra A, and a left A-module M, an R-derivation D of A in M is an R-linear map $D : A \to M$ satisfying $D(ab) = a.Db + b.Da$ (the Leibniz rule). If A, B are two R-algebras and M is an $A \otimes_R B$-module, and thus also an A-module and a B-module, there is a natural isomorphism $(\mathrm{Der}_R(A, M) \otimes B) \oplus (A \otimes \mathrm{Der}_R(B, M) \cong \mathrm{Der}_R(A \otimes_R B, M))$.

Given a commutative ring R, a map $\phi : A \to B$ of R-modules, and a B-module N, the map ϕ induces an R-module homomorphism $\phi_0 :\mathrm{Der}_R(B,N) \to \mathrm{Der}_R(A,N)$ whose kernel is $\mathrm{Der}_A(B,N)$, regarding B as an A-module in the obvious way. Thus we get an exact sequence of A-modules $0 \to \mathrm{Der}_A(B,N) \to \mathrm{Der}_R(B,N) \to \mathrm{Der}_R(A,N)$ of A modules. In particular, given a morphism $\phi : X \to Y$ of affine varieties, we have its *differential* $d\phi_x : T_x(X) \to T_{\phi x}(Y)$, a linear map on $\mathbf{k}$-vector spaces, for each $x \in X$; given another morphism of varieties $\psi : Y \to Z$ one has the chain rule $d(\psi \circ \phi)_x = d\psi_{\phi x} \circ d\phi_x$; if ϕ is an isomorphism, so is $d\phi_x$ for any x, and the differential of the identity morphism is the identity map (p. 58). As a special case, if G is an algebraic group and X is a G-variety, then the action of any $g \in G$ defines an isomorphism from the tangent space T_xX of X at any point x to the tangent space $T_{g.x}X$ of X at $g.x$.

It is immediate from the definition that any $\mathbf{k}$-derivation D of $\mathbf{k}[X]$ in $\mathbf{k}_x$ sends any constant function or function in M_x^2 to 0. Hence it induces and is completely determined by a $\mathbf{k}$-linear map $\lambda(D)$ from the quotient M_x/M_x^2 to $\mathbf{k}$; conversely, any such map $\lambda(D)$ corresponds to a unique derivation D (Lemma 4.1.4, p. 58).

We can also extend derivations uniquely to certain rings of regular functions: given a $\mathbf{k}$-derivation D of $\mathbf{k}[X]$ in $\mathbf{k}_x$, it extends uniquely to a regular function in $\mathcal{O}_x$ (that is, function on X regular at x) via the quotient rule $D(g/h) = (h.Dg - g.Dh)/h^2$; note also that the quotient $\mathcal{O}_x/\mathcal{M}_x$ by its unique maximal ideal $\mathcal{M}_x$ of regular functions vanishing at x is also isomorphic to $\mathbf{k} = \mathbf{k}_x$. We also see that *if* ϕ *is an isomorphism from* X *onto an affine open subvariety of* Y*, then* $d\phi_x$ *is an isomorphism from* $T_x(X)$ *onto* $T_{\phi x}(Y)$ *for any* $x \in X$ (Lemma 4.1.6, p. 59). Hence we can extend the definition of tangent space to an arbitrary variety X: if U, V are two affine open neighborhoods of $x \in X$ with $V \subset U$ then there is a canonical identification $T_x U \cong T_x V$, so that we can take $T_x X$ to be $T_x U$ for any open affine neighborhood U of x.

Definition 4.1.7, p. 59

We say that $x \in X$ is *simple*, or *nonsingular*, or X is *smooth* at x, if the dimension $\dim T_x X = \dim X$. X is *smooth*, or *nonsingular*, if all of its points are smooth.

There is a handy way to reformulate the notion of derivation. Given as usual a commutative ring R and an R-module A we have the multiplication map $m : A \otimes_R A$ defined by $m(a \otimes b) = ab$. Letting I be the kernel of m we note that the quotient $(A \otimes A)/I$ is naturally isomorphic to A.

Definition 4.2.1, p. 60

The *module of differentials* $\Omega_{A/R}$ is the quotient I/I^2, viewed as an A-module thanks to the identification $A \otimes A/I \cong A$. The image of $a \otimes 1 - 1 \otimes a$ in $\Omega_{A/R}$ is denoted da or $d_{A/R}a$.

Theorem 4.2.2, p. 61

For every A-module M the map $\Phi : \hom_A(\Omega_{A/R}, M) \to \mathrm{Der}_A(M)$ sending ϕ to $\phi \circ d$ is an A-module isomorphism.

Proof.

Clearly Φ is an A-module map, which is injective since the da generate $\Omega_{A/R}$. Letting $D \in \mathrm{Der}_R(A, M)$, define an R-linear map $\psi : A \otimes A \to M$ via $\psi(a \otimes b) = bDa$; then $\psi(x, y) = m(x)\psi(y) + m(y)\psi(x)$, whence ψ vanishes on I^2 and defines an R-linear map from $\Omega_{A/R}$ to M. Since $\psi(a \otimes 1 - 1 \otimes a) = Da$ we have $\Phi\psi = D$ and Φ has the required properties. □

Let A be an R-algebra of the form $R[x_1, \ldots, x_m]/(f_1, \ldots, f_n)$. Let t_i be the image of x_i in A, write $t = (t_1, \ldots, t_m)$, and as above denote partial differentiation with respect to x_i by D_i. Then a straightforward calculation essentially repeating one given above yields

Lemma 4.2.4, p. 61

The dt_i generate $\Omega_{A/R}$. The kernel of the homomorphism $\phi : A^m \to \Omega_{A/R}$ with $\phi(e_i) = dt_i$, e_i the ith unit coordinate vector, is the submodule generated by the elements $\sum_{i=1}^{n} D_i f_j(t) e_i$ for $1 \leq j \leq n$.

For an affine variety X we can regard the $\mathbf{k}[X]$-module $\Omega_{\mathbf{k}[X]/\mathbf{k}}$ as the tangent object of X promised above, though it is too large to be an actual tangent space.

In particular, take R to be our basefield $\mathbf{k}$ and A to be the coordinate ring $\mathbf{k}[X]/I$, where $I = (f_1, \ldots, f_n)$ is the ideal of definition of the affine variety $X \subset \mathbf{k}^m$. Let J be the Jacobian matrix of the f_i, so that the ijth entry of J is the partial derivative $\partial f_i/\partial x_j$ of f_i with respect to the jth variable x_j. Evaluating J at any point $x \in \mathbf{k}^m$ we get a matrix $J(x)$ of elements of $\mathbf{k}$, whose corank (the number m of variables minus the rank) equals the dimension of the tangent space of X at x; note that this space is now given to us as a quotient rather than a submodule of $\mathbf{k}^m$, as it was in the heuristic introduction, but its dimension over $\mathbf{k}$ at any x is the same as it was before.

I now enlarge the domains of definition of the above derivations to fields. Let F be a field and $E \subset E'$ be two finitely generated field extensions of F. By above I have an exact sequence of groups $0 \to \mathrm{Der}_{E'}(E, E) \to \mathrm{Der}_F(E, E) \to \mathrm{Der}_F(E', E)$; this is also an exact sequence of E-vector spaces, where the E-action comes form the second arguments. Then we get an exact sequence $0 \to \hom_E(\Omega_{E/E'}, E) \to \hom_E(\Omega_{E/F}, E) \to \hom_{E'}(\Omega_{E'/F}, E)$, yielding an exact sequence of E-vector spaces
$0 \to \hom_E(\Omega_{E/E'}, E) \to \hom_E(\Omega_{E/F}, E) \to \hom_E(E \otimes_{E'} \Omega_{E'/F}, E)$, since $\hom_E(E \otimes_{E'} \Omega_{E'/F}, E)$ is naturally isomorphic to $\hom_{E'}(\Omega_{E'/F}, E)$. It is easy to check that these vector spaces are finite-dimensional. Dualizing we get an exact sequence of E-vector spaces
$E \otimes_{E'} \Omega E'/F \to \Omega_{E/F} \to \Omega_{E/E'} \to 0$; denote the leftmost map here by α.

Now recall that E is called *separable algebraic* over F if for each $x \in E$ there is a polynomial $f \in f[T]$, or equivalently an irreducible polynomial $f \in f[T]$, over F without multiple roots such that $f(x) = 0$. If the characteristic of F is 0, then every algebraic extension of F is separable algebraic.

Lemma 4.2.7, p. 63

If E is separable algebraic then the map α above is injective.

Proof.

From the above discussion we see that the injectivity of α is equivalent to the surjectivity of the homomorphism $\mathrm{Der}_F((E, E') \to \mathrm{Der}_F(E', E)$, or equivalently to the assertion that every F-derivation of E' in E extends to an F-derivation of E in E. To prove this it is enough to deal with a simple extension $E = E'(x) \cong E'[T]/(f)$, where f is an irreducible polynomial with $f'(x) \neq 0$. Let $D \in \mathrm{Der}_F(E', E)$. If $g = \sum_{i \geq 0} a_i T^i \in E'[T]$ define $Dg \in E[T]$ via $Dg = \sum (Da_i) T^i$. Then D extends to an F-derivation D' of E in E with $D'x = a$ if and only if $f'(x)a + (Df)(x) = 0$. Since $f'(x) \neq 0$ this equation has a unique solution and the lemma follows. □

Lemma 4.2.8, p. 63

If $E = F(x)$ then $\dim_E \Omega_{E/F} \leq 1$. We have $\Omega_{E/F} = 0$ if and only if E is separably algebraic over F.

The proof is an easy exercise.