Lecture 10-13: Wrapping up commutative groups

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I will wrap up Chapter 3, giving an account of diagonalizable groups and showing that a connected commutative group consisting of semisimple (resp. unipotent) elements is a torus (resp. a vector group).

Definition 3.2.1, p. 43

A *diagonalizable group* is one isomorphic to a closed subgroup of a torus.

Before discussing diagonalizable groups in general, I will study tori in more detail.

Example

Let $x_1 = \chi_1(x), \ldots, x_n = \chi_n(x)$ be the diagonal entries of $x \in D_n$. Recall that a *character* of a group G is a homomorphism $G \to G_m$; note that the set $X^*(G)$ of characters of G is a group under multiplication. Recall *Dedekind's theorem*, which asserts that the characters of G are linearly independent as **k**-valued functions on G; this is easily proved by considering a dependence relation with as few terms as possible and then using the definition of character to produce such a relation with fewer terms, a contradiction. For $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ we have a character of D_n sending the element x above to $x_1^{a_1} \ldots x_n^{a_n}$. Since these functions span all of $\mathbf{k}[D_n]$ over **k**, it follows by Dedekind that $X^*(D_n) \cong \mathbb{Z}^n$.

Example

By contrast, a vector group G (isomorphic to G_a^n for some n) has no nontrivial characters at all. The homomorphisms from G into G_a, called *additive functions* (p. 49), are the same as they are in linear algebra, that is, they are just $\mathbf{k}$-vector space maps from $G \cong \mathbf{k}^n$ to $\mathbf{k}$. Monomials in these maps (but not just the maps themselves) furnish a $\mathbf{k}$-basis of $\mathbf{k}[G]$.

Theorem 3.2.3, p. 43

The following are equivalent:

- G is diagonalizable.
- $X^*(G)$ is a finitely generated abelian group whose elements span $\mathbf{k}[G]$.
- Any rational representation of G is a direct sum of one-dimensional representations.

Proof.

If G is a closed subgroup of D_n then $\mathbf{k}[G]$ is a quotient of $\mathbf{k}[D_n]$; since any character of D_n restricts to a character of G, we see that the nontrivial restrictions to G of the characters of D_n span $\mathbf{k}[G]$ and that they form a basis of $X^*(G)$. Hence $X^*(G)$ is a finitely generated abelian group. Now suppose that $X = X^*(G)$ is finitely generated and abelian and spans $\mathbf{k}[G]$ and let $\phi : G \to GL(V)$ be a rational representation. Then we must have that each $\phi(x)$ is a finite linear combination $\sum_{\chi \in X} A_\chi \chi(x)$, where the A_χ are suitable linear maps on V, since the $\chi(x)$ span $\mathbf{k}[G]$. Finally, if the third property holds, then so too does the first one, embedding G in D_n and looking at the action of G on $V = \mathbf{k}^n$. □

Proof.

The relation $\phi(xy) = \phi(x)\phi(y)$, together with Dedekind's theorem, implies that $A_\chi A_\psi = A_\chi = A_\psi$ if $\chi = \psi$ and $A_\chi A_\psi = 0$ otherwise; also $\sum_\chi A_\chi = 1$. Letting V_χ be the image of A_χ it follows that V is the direct sum of the V_χ and that every $x \in G$ acts on V_χ by the scalar $\chi(x)$, as claimed. $\square$

Now let G be a diagonalizable group with character group $X^*(G)$. As a finitely generated abelian group $A = X^*(G)$ is isomorphic to the direct sum $\mathbb{Z}^n \otimes M$ for some finite abelian group M. Define the *group algebra* $\mathbf{k}[A]$ to be the set of finite formal linear combinations $\sum_{a \in A} k_a e(a)$ with coefficients $k_a \in \mathbf{k}$, multiplying by the distributive law together with the rule $e(a)e(b) = e(a+b)$ for $a, b \in A$. One easily checks that $\mathbf{k}[A]$ has the structure of a linear algebraic group with coordinate ring isomorphic to $\mathbf{k}[T_1, T_1^{-1} \ldots, T_n, T_n^{-1}] \otimes \mathbf{k}[M]$, defining $\mathbf{k}[M]$ analogously to $\mathbf{k}[A]$.

It follows that $\mathbf{k}[G]$, since it is spanned over $\mathbf{k}$ by $X^*(G)$, is isomorphic to the coordinate ring of $\mathbf{k}[A]$, whence G is isomorphic to $\mathbf{k}[A]$, or to the direct product $D_n \times M$. We deduce that

Corollary 3.2.7, p. 45

Any diagonalizable group G is the direct product of a torus and a finite abelian group, with the order of the latter prime to the characteristic of $\mathbf{k}$, if this is positive. G is a torus if and only if it is connected, or if and only if $X^*(G)$ is free abelian.

This is immediate, noting that $\mathbf{k}$ has no pth roots of 1 other than 1 if it is has prime characteristic p.

Next I give a result showing how diagonalizable groups fit inside larger groups.

Proposition 3.2.8, p. 45: rigidity of diagonalizable groups

Let G, H be diagonalizable groups and let V be a connected variety. Assume we are given a morphism of varieties $\phi : V \times G \to H$ such that for any $v \in V$ the map $v \to \phi(v, x)$ defines a homomorphism of algebraic groups from G to H. Then $\phi(v, x)$ is constant in v.

Indeed, for any $\psi \in X^*(H)$ we have $\psi(\phi(v, x)) = \sum_{\chi \in X^*(G)} f_{\chi,\psi}(v)\chi(x)$ with $f_{\chi,\psi} \in \mathbf{k}[V]$. Dedekind implies that $f_{\chi,\psi}(v) = 1$ for one χ and 0 for the others, whence $f^2_{\chi,\psi} = f_{\chi,\psi}$. Then the connectedness of V forces for each fixed ψ that $f_{\chi,\psi} = 1$ for one χ and 0 for the others; the result follows.

Corollary 3.2.9, p. 46

Let H be a diagonalizable subgroup of a group G. Then the centralizer $Z_G(H)$ of H in G has finite index in its normalizer $N_G(H)$ and $Z_G(H), N_G(H)$ have the same identity component.

It is immediate that $Z_G(H), N_G(H)$ are closed subgroups with $Z_G(H)$ normal in $N_G(H)$. Then the result follows from the proposition applied to the identity component $V = N_G(H)^0$ of $N_G(H)$ and ϕ the morphism from $V \times H$ to H sending (x, y) to xyx^{-1}.

It follows from above results that a commutative group consisting of semisimple elements is diagonalizable and thus the direct product of a torus and a finite abelian group. What about a commutative unipotent group, assuming further that its elements have order dividing p if the characteristic p of $\mathbf{k}$ is prime?

Theorem 3.4.7, p. 54

Any such group G is a vector group G_a^n if the characteristic p of $\mathbf{k}$ is 0 and the direct product of a vector group and a product of copies of $\mathbb{Z}/p$ if p is positive. In particular, G is a vector group if and only if it is connected.

I refer to pp. 49–54 of the text for the proof. The argument is elementary but quite elaborate, requiring a cohomological calculation with polynomials and proving along the way that $\mathbf{k}[G]$ is generated as a $\mathbf{k}$-algebra by the additive functions on G.

As an immediate corollary we get

Theorem 3.4.9, p. 55

Up to isomorphism, the only connected groups of dimension one are G_a and G_m.

We already know that all such groups are commutative and are either unipotent or consist entirely of semisimple elements. Then the result follows at once from the previous one.