Lecture 10-11: Jordan decomposition continued and commutative groups

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I now tie the Jordan decomposition to algebraic groups. Given such a group G and $g \in G$, we have seen that right translation $\rho(g)$ by g on $\mathbf{k}[G]$ (acting via the recipe $(\rho(g)f)(x) = f(xg)$) is locally finite, so we have a unique Jordan decomposition $\rho(g) = \rho(g)_s \rho(g)_u$

Theorem 2.4.8, p. 34: Jordan decomposition in G

Given $g \in G$ there are unique $g_s, g_u \in G$ with $\rho(g_s) = \rho(g)_s, \rho(g_u) = \rho(g)_u, g = g_s g_u = g_u g_s$. If $\phi : G \to G'$ is a homomorphism, then $\phi(g_s) = \phi(g)_s, \phi(g_u) = \phi(g)_u$. If $G = GL_n(\mathbf{k})$, then $g = g_s g_u$ is the Jordan decomposition of G defined previously.

Proof.

Since $\rho(g)$ is an algebra automorphism of $\mathbf{k}[G]$, it commutes with addition and multiplication in $\mathbf{k}[G]$, whence by a property of the Jordan decomposition proved last time so do $\rho(g)_s, \rho(g)_u$. Given a $\mathbf{k}$-algebra homomorphism of $\mathbf{k}[G]$ into k, corresponding to an element of G, its compositions with $\rho(g)_s \rho(g)_u$ thus also correspond to commuting elements g_s, g_u of G, respectively, with $g = g_s g_u$. Uniqueness follows since ρ is faithful on G. Given a homomorphism ϕ it factors as a surjective homomorphism onto its image followed by an inclusion of this image into G'. Again previous properties of the Jordan decomposition yield the second assertion. The third one follows similarly. □

We say that $g \in G$ is *semisimple* if $g = g_s$ and similarly that $g \in G$ is *unipotent* if $g = g_u$. It follows at once that $g \in G$ is semisimple if and only if $\phi(g)$ is semisimple as a matrix for any homomorphism $\phi : G \to GL(n, \mathbf{k})$; similarly for unipotent elements. We also deduce a significant constraint on closed (or equivalently algebraic) subgroups of $GL(n, \mathbf{k})$: any such subgroup must contain the semisimple and unipotent parts of all of its elements.

Proposition 2.4.12, p. 36

If G is a subgroup of GL_n consisting of unipotent matrices, then there is $x \in GL_n$ with $xGx^{-1} \subset U_n$, the subgroup of upper triangular unipotent matrices.

We call any such subgroup *unipotent*; by above results unipotent algebraic groups are exactly those consisting of unipotent elements, or conjugate to a subgroup of U_n.

We know that G acts linearly on $V = \mathbf{k}^n$. If there is a proper subspace W of V stabilized by G, then the result at once by induction on $\dim V$. If there is no such subspace, then G acts irreducibly on V. Then it is well known that linear combinations of matrices in G fill out all of M_n, the algebra of $n \times n$ matrices over $\mathbf{k}$. Any matrix in G has trace n, whence the trace of $(1 - g)h$ is 0 for all $g, h \in G$, whence also for $g \in G, h \in M_n$. Since the trace form on M_n sending any ordered pair(x, y) of matrices to the trace of their product xy is nondegenerate, so that no nonzero matrix is orthogonal to every other under this form, this forces $G = 1$ and the result is trivial.

Next I show that unipotent groups act on affine varieties with closed orbits.

Proposition 2.4.14, p. 37

If G is unipotent and X is an affine G-space, then all orbits of G on X are closed.

Proof.

Let O be a G-orbit. Replacing X by the closure $\overline{O}$ we may assume that O is dense in X and then we know that O is also open in X. Letting Y be the complement of O in X, we have that G acts locally finitely on the ideal of functions in $\mathbf{k}[X]$ vanishing on Y, whence if Y is nonempty there is a nonzero function f vanishing on Y and fixed by G, which must be constant on O and thus on X. This is a contradiction, forcing $O = X$, as desired. □

Skipping the rest of Chapter 2 (pp. 37-41) we proceed to Chapter 3, treating commutative algebraic groups.

Theorem 3.1.1, p. 42

Let G be a commutative algebraic group. The sets G_s, G_u of semisimple and unipotent elements of G are closed subgroups and G is isomorphic to their direct product. If G is connected so are G_x and G_u.

Proof.

We may assume G is a closed subgroup of some GL_n, Since the product of two commuting semisimple (resp. unipotent) elements of G is again semisimple (resp. unipotent), we see that G_s, G_u are subgroups with product G. Since a commuting family of semisimple matrices is conjugate to a subset of the set D_n of diagonal matrices, we can arrange things so that G lies in the the subgroup T_n of upper triangular matrices and G_s is its intersection with the subgroup D_n of diagonal matrices, whence G_s is closed; likewise $G_u = G \cap U_n$ is closed. The uniqueness of the Jordan decomposition shows that the product map $\pi : G_s \times G_u \to G$ is an isomorphism of abstract groups, while the map sending g to g_s picks out the diagonal entries of g, so is a morphism. Hence π is an isomorphism of algebraic groups, as claimed. Finally, if G is connected then so are its images G_s, G_u under the projection maps. □

I now specialize to (algebraic) groups of dimension one.

Proposition 3.1.3, p. 42

Any group G of dimension one is commutative, and either equal to G_s or G_u. If $G = G_u$ and $\mathbf{k}$ has characteristic $p > 0$, then all elements of G have order dividing p.

Proof.

Fix $g \in G$ and let $\phi : G \to G$ send x to $g^{-1}xg$. The closure $\overline{\phi G}$ is then an irreducible closed subset of G; since G has dimension one it must be a singleton or all of G. If it is all of G then the image ϕG, being open in G, must contain all but finitely many elements of G. Viewing G as a closed subgroup of some GL_n, this forces the set of characteristic polynomials $\det(T.1 - x)$ to be a finite set as x runs over G. The connectedness of G then implies that every $x \in G$ has characteristic polynomial $(T-1)^n$ and G is unipotent, forcing $G \subset U_n$ after replacing G by a conjugate. Letting $G_1 = [G, G]$, the commutator subgroup of G, and $G_2 = [G_1, G_1]$, etc., we must have $G_n = 1$ and yet G_1 is either 1 or all of G, forcing $G_1 = 1$ and G is commutative. The product $G = G_x \times G_u$ forces $G = G_s$ or $G = G_u$. Finally, if $G = G_u$ and $p > 0$, then the set G^{p^k} of p^k-th powers of elements of G is a subgroup, which must be trivial if $p^k > n$, forcing $G^p = 1$, as desired. $\square$

I conclude with the definitions of torus (different from the one a topologist or geometer would use) and vector group.

Definitions 3.2.1, p. 43 and 3.4.1, p. 51

An (algebraic) *torus* is a group isomorphic to $D_n \cong G_m^n$ for some n; recall that $G_m = \mathbf{k}^*$ is the multiplicative group of nonzero elements of $\mathbf{k}$. Likewise a *vector group* is one isomorphic to $\mathbf{k}^n \cong G_a^n$, where $G_a = \mathbf{k}$, considered as an additive group.