Lecture 3-8

We begin by dotting some i's in the Weyl character formula. Applying a simple reflection s_α to the half-sum ρ of the positive roots, we find that the simple root α is replaced by its negative in the half sum, while the other positive roots are permuted. Hence $s_\alpha \rho - \rho - \alpha, 2\rho \cdot \alpha/\alpha \cdot \alpha = 1$, and we see that ρ is the sum of the fundamental dominant weights, having dot product 1 with the dual of any simple root. It follows that if λ is dominant, then $\lambda+\rho$ is *strictly* dominant, having positive dot product with all positive roots. If $w(\lambda+\rho)$ is dominant for any element w of the Weyl group W, then $w(\lambda+\rho)$ is not orthogonal to any root, so is strictly dominant, whence w must send the set of positive roots to itself, whence it must permute the simple root and act by an automorphism of the Dynkin diagram. But we have already seen that the Weyl group element that does this is the identity, whence $w = 1$. Hence all terms $(\det w)e^{w(\lambda+\rho)}$ and $(\det w)e^{w\rho}$ in the numerator and denominator of the Weyl character formula are distinct; there is no cancellation in this formula.

Our next task is to use this formula to compute the dimension $\dim L_\lambda$ of the irreducible module of highest weight λ, assumed dominant integral. There is a homomorphism v from the ring R of formal characters to the integers sending any e^μ to 1, so at first we are tempted to apply v to the numerator and denominator of the formula for ch L_λ; unfortunately both have image 0 under this homomorphism. We get around this by in effect applying L'Hopital's Rule. Given a positive root α, the linear map ∂_α from R to itself sending any e^μ to $(\mu \cdot \alpha)e^\mu$ is a derivation (it satisfies the product rule). Now let ∂ be the product of all the ∂_α as α runs over the positive roots. Apply ∂ to both sides of the character formula $\sum_{w\in W}(\det w)e^{w\rho} = \sum_{w\in W} e^{w(\lambda+\rho)}$, and then apply v to both sides. Since $\Delta = \sum_{w\in W}(\det w)e^{w\rho} = \prod_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2})$ and each factor $e^{\alpha/2} - e^{-\alpha/2}$ is sent to 0 by v, we see that the only nonzero contribution to the left side comes from $v(\partial\Delta)v(\text{ch } L_\lambda) = v(\partial\Delta)\dim L_\lambda$, while the right side is $v\partial(\sum_{w\in W}(\det w)e^{w(\lambda+\rho)})$. Now for every term $(\det w)e^{w(\lambda+\rho)}$ occurring in the numerator of the character formula, the product of the dot product $w(\lambda+\rho)\cdot\alpha$ changes by a sign if w is replaced by $s_\alpha w$ for α simple, since as usual s_α sends exactly one positive root to its negative and permutes the others. Hence $\partial(\det w)e^{w(\lambda+\rho)} = \partial e^{\lambda+\rho} = e^\lambda \prod_{\alpha\in\Phi^+}((\lambda+\rho)\cdot\alpha)$ for all $w \in W$, and similarly for $\partial(\det w)e^{w\rho}$. Applying v, we get $|W|\prod_{\alpha\in\Phi^+}((\lambda+\rho)\cdot\alpha)$ in the numerator and $|W|\prod_{\alpha\in\Phi^+}(\rho\cdot\alpha)$ in the denominator. Hence $\dim L_\lambda = \prod_{\alpha\in\Phi^+}((\lambda+\rho)\cdot\alpha)/(\rho\cdot\alpha)$; this is the *Weyl dimension formula*.

To apply this formula it is customary to replace all positive roots α by their dual roots $2\alpha/(\alpha\cdot\alpha)$; then all factors in the numerator and denominator are integers. Now we recall that the set of dual positive roots is a positive subsystem for the dual root system to Φ. By running through all positive roots in this subsystem, writing each as a combination of dual simple roots, and writing $m_i \in \mathbb{N}$ for the dot product of λ with the ith dual simple root, we get a homogeneous polynomial in the variables m_i+1 representing the dot products of $\lambda+\rho$ with the dual simple roots. For example, in type G_2, if the simple roots are written as $\alpha = (0,1,-1)$ and $\beta = (1,-2,1)$, then the positive roots are $\alpha, \beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, 2\beta+3\alpha$. Replacing every positive root by its dual and writing α', β' for the duals of α, β, we find that the dual positive roots are $\alpha', \beta', \alpha'+\beta', \alpha'+2\beta', '\alpha+3\beta', 2\alpha'+3\beta'$. Now let m_1, m_2

be the respective dot products of λ with α', β'. Applying the Weyl dimension formula, we find that the dimension of L_λ is given by

$$\frac{(m_1+1)(m_2+1)(m_1+m_2+2)(m_1+2m_2+3)(m_1+3m_2+4)(2m_1+3m_2+5)}{5!}$$

In particular, if $m_1 = 1, m_2 = 0$, we get 7 for this dimension and if $m_1 = 0, m_2 = 1$, we get 14 for this dimension. In the latter case, we can recognize λ as the highest root, so L_λ is the adjoint representation and we recover that the dimension of a Lie algebra L of type G_2 is 14. In the former case, we get the smallest nontrivial representation of L. It turns out that L acts by skew-adjoint transformations with respect to a symmetric bilinear form on this representation, so L embeds in $\mathfrak{so}(7)$, the Lie algebra of type B_3. By contrast, the unique largest exceptional simple Lie algebra L' of type E_8 admits no nontrivial finite-dimensional module of dimension less than 248, the dimension of L' itself.