Lecture 3-6

Last time I introduced the *formal character* ch N_λ of any subquotient N_λ of a Verma module M_λ as the formal sum $\sum_\mu \dim(N_\lambda)_\mu e^\mu$, where the sum runs over all the weights μ of N_λ; this sum can be infinite but still makes sense, since all weight spaces of N_λ, like those of M_λ, are finite-dimensional. Let me way a few words about the use of the term "character" in this context. Usually this term refers to the trace of the matrix attached to an element of a group by a representation of that group. Working for simplicity over the basefield $\mathbb{C}$, we have the aforementioned exponential map taking any Lie algebra L of matrices to a corresponding Lie group G; it sends a matrix M to $e^M = \sum_{i=0}^\infty M^i/i!$. If a representation W of L has a nonzero μ-weight space W_μ, then the matrices h in any Cartan subalgebra H of L act on W_μ with trace $\dim W_\mu \mu(h)$, whence the matrices in the corresponding Cartan subgroup e^H act on W_μ with trace $\dim W_\mu e^{\mu(h)}$. If μ is finite-dimensional, then the sum of all terms $\dim W_\mu e^{\mu h}$ is the trace of h on W; if W is infinite-dimensional, then we have to worry about convergence issues for the sum defining ch W, but at least this motivates our definition of the sum and product of any two expressions $\sum_\mu a_\mu e^\mu$ and $\sum_\mu b_\mu e^\mu$; note that the weights μ occurring for example in any highest weight module will be such that the product of any two such sums running over the weights in such a module is well defined.

Now it is easy to see that given a chain of submodules $N_0 \subset N_1 \subset N_2 \subset \ldots$ of a highest weight module N, the character ch N of N is the sum of the characters of the subquotients N_i/N_{i-1} for $i \geq 1$. Last time we saw that all irreducible subquotients of any Verma module M_λ are themselves irreducible highest weight modules L_μ and that there are only finitely many possibilities for μ (any μ occurring in this way must lie in the λ-coset of the weight lattice of L in $\mathbb{R}^n$ and have $\mu+\rho$ of the same square length at $\lambda+\rho$ where ρ is the half sum of the positive roots. Since the weight spaces of N_λ are finite-dimensional, it follows that M_λ in fact admits a *finite* chain of submodules whose subquotients all take the form L_μ for some μ lying below λ in the partial order and satisfying the condition above. If we now fix λ and let $\mu_1 = \lambda, \ldots, \mu_n$ denote all μ satisfying this condition, then we can order the μ in such way that ch M_{μ_i} is the sum of ch L_{μ_i} and a nonnegative integral combination of ch L_{μ_j} for $j > i$ (so that in particular $M_{\mu_n} = L_{\mu_n}$ is irreducible. This gives us an upper triangular system of linear equations relating the ch M_{μ_i} to the ch L_{μ_i} with integer entries and ones on the main diagonal. Any such system can be inverted, so we deduce that *ch L_{μ_i} is an integral linear combination $\sum n_j$ ch M_{μ_j} of ch M_{μ_j} in which ch M_{μ_i} appears with coefficient $n_i = 1$*; the other coefficients can be negative integers in general.

Now for the first time we introduce our key hypothesis that λ is dominant integral, so that L_λ is the unique finite-dimensional irreducible module with highest weight λ. Consider a typical term ch M_μ appearing in the combination for ch L_λ. By the PBW Theorem we may write ch M_μ as $e^\mu \prod_{\alpha\in\Phi^+}(1 + e^{-\alpha} + e^{-2\alpha} + \ldots) = e^\mu \prod_{\alpha\in\Phi^+} \frac{1}{1-e^{-\alpha}}$. Multiplying the numerator and denominator of the α term in this product by $e^{\alpha/2}$ and using the definition of ρ, we get the more symmetric expression $\frac{e^{\mu+\rho}}{\prod(e^{\alpha/2}-e^{-\alpha/2})}$. Now we bring in the last piece of the puzzle: ch L_λ is invariant under the natural action of the Weyl group W, where $w \in W$ sends anye^μ to $e^{w\mu}$. Applying any simple reflection s_β to the product $\Delta = \prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})$, we find that the term $e^{\beta/2} - e^{-\beta/2}$ is replaced by its negative, while the other terms, corresponding to positive roots other than β, are permuted; hence the product is sent to its negative. Hence any $w \in W$, when applied to this product, sends it to $\det w$ times itself, where $\det w$ is the determinant of w regarded as a linear transformation of the Euclidean space H^* containing the roots.

The formula $\text{ch } L_\lambda = \sum_\mu \text{ch } M_\mu$, combined with the W-invariance of $\text{ch } L_\lambda$ and the skew invariance of the denominator Δ, guarantees that the numerator $\sum n_i e^{\mu_i+\rho}$ is skew-invariant under W; the coefficient of any term e^ν in it equals $\det w$ times the coefficient of $e^{w\nu}$ in it. But now the coefficient of $e^{\lambda+\rho}$ is one, so the coefficient of $e^{w(\lambda+\rho)}$ is $\det w$. Now finally the only $e^{\mu+\rho}$ appearing in this sum with μ dominant integral is $\mu = \lambda$, since other dominant integral weights $\mu+\rho$ below $\lambda+\rho$ have shorter length than $\lambda+\rho$, so our final grand conclusion is the Weyl character formula: $\text{ch } L_\lambda = \frac{\sum_{w\in W} (\det w)\, e^{w(\lambda+\rho)}}{\Delta}$. In particular, taking $\lambda = 0$, we find that L_λ is the trivial representation, having weight 0 with multiplicity 1 and no other weights, so we get a formula for the *Weyl denominator* Δ; it is $\sum_{w\in W} (\det w) e^{w\rho}$.