Lecture 3-4

We now look at other representations of classical Lie algebras (besides the defining representations). Recall that the tensor product $\otimes_i V_i$ of two or more modules V_i over a Lie algebra L also carries the structure of an L-module. In the case of tensor powers $T^nV = V^{\otimes n}$ of a single L-module V, certain natural quotients of T^nV also admit an L-module structure. Specifically, if we pass to the nth *symmetric power* S^nV, obtained from T^nV by moding out by the relation $v \otimes w = w \otimes v$ for $v, w \in V$ (that is, by identifying two pure tensors in T^nV whenever one is obtained form the other by interchanging two terms), then the defining relation is preserved by the action of L, so L continues to act on S^nV. Similarly, if instead we impose the relation $v \otimes w = -w \otimes v$ for $v, w \in V$, then we get the nth *exterior* power $\wedge^n V$, which is also an L-module whenever V is. Note that the dimension of $\wedge^n V$ is the binomial coefficient $\binom{m}{n}$, where m is the dimension of V, so $\wedge^n V = 0$ for $n > m$; by contrast, S^nV has dimension $\binom{m+n-1}{n}$, which grows arbitrarily large as n increases.

Now it turns out that if V is the defining representation of a classical Lie algebra L, then the powers $S^nV, \wedge^n V$ are "irreducible modulo the obvious relations". More precisely, all symmetric and exterior powers S^nV, T^nV are irreducible over L if $L = \mathfrak{sl}(n, K)$, with the highest weight of S^nV equal to n times the highest weight of V itself. If L is orthogonal (so of type B or D), then S^2V has a nonzero vector sent to 0 by L, corresponding to the symmetric bilinear form preserved by Int L; accordingly, we may view $S^{n-2}V$ as a submodule of S^nV and the quotient $S^nV/S^{n-2}V$ turns out to be irreducible; its highest weight is n times the highest weight of V. The exterior powers $\wedge^n V$ remain irreducible (or 0) for all n; there is no skew-symmetric form preserved by Int L. In type C, we have the opposite situation; there is a skew-symmetric bilinear form preserved by Int L; accordingly, $\wedge^2 V$ admits an Int L-invariant vector (sent to 0 by L) and $\wedge^{n-2} V$ identifies with a submodule of $\wedge^n V$ if $2 \le n \le m = (1/2)\dim V$ and $\wedge^n V / \wedge^{n-2} V$ is irreducible. Its highest weight has coordinates $(1, \dots, 1, 0, \dots, 0)$, where n ones appear. In types A and C, one can start from V, apply a purely linear algebraic construction generalizing the symmetric and exterior powers, and get all the irreducible finite-dimensional L-modules; this is the approach followed by Weyl in his famous 1939 book on the classical groups. In types B and D, the starting point of V is not enough; one needs the so-called *half-spin* representation in type B and two half-spin representations in type D. Their highest weights are $((1/2, \dots, 1/2)$ in type B_n and $(1/2, \dots, 1/2), (1/2, \dots, 1/2, -1/2)$ in type D_n; in both cases all weights are W-conjugates of the highest weight and all multiplicities are one, so that the dimension of the half-spin representation in type B_n is 2^n, while the dimensions of the half-spin representations in type D_n are both 2^{n-1}. As mentioned last time, all of these representations carry actions of the appropriate spin group, but not the corresponding orthogonal group.

Returning now to a general semisimple Lie algebra L with Cartan subalgebra H, let $\lambda \in H^*$ be any weight (not necessarily integral). Let I be the left ideal of the enveloping algebra U of L arising in the definition of highest weight representation, so that I is generated by all positive root space L_α of L together with $h - \lambda(h)$ for all $h \in H$. The quotient U/I is called a *standard cyclic module* in the text; nowadays it is more commonly known as a *Verma module* and denoted M_λ. Any L-module generated by a single highest weight vector of weight λ is a quotient of M_λ; the unique irreducible such module is denoted L_λ (where this notation replaces our earlier one V^λ).

Now the weights of M_λ all take the form $\lambda - \nu$, where ν is a sum of simple roots; thus if a sum of sufficiently many simple roots is added to any weight of M_λ, the resulting weight does not occur in M_λ. It follows that any subquotient S of M_λ has a weight that is maximal in the partial order, and thus a highest weight vector. Thus an irreducible subquotient of M_λ must take the form L_μ for some weight μ lying below λ. But now I claim that there are only finitely many possibilities for μ. To prove this, we introduce an analogue in U of the Casimir element that we used in the proof of Weyl's Theorem. Let $h_1, \ldots, h_r$ be an orthonormal basis of H with respect to the Killing form κ. For each positive root α, choose $x_\alpha \in L_\alpha, z_\alpha \in L_{-\alpha}$ so that $\kappa(x_\alpha, z_\alpha) = 1$. Note that z_α is not the same as our usual $y_\alpha \in L_{-\alpha}$ corresponding to x_α; this time we have $[x_\alpha, z_\alpha] = t_\alpha$, the element of H identified with $\alpha \in H^*$. Now the element $c = \sum_{i=1}^r h_i^2 + \sum_{\alpha\in\Phi^+}(x_\alpha z_\alpha + z_\alpha x_\alpha)$ of U commutes with all $x \in L$, by the same argument as for the original Casimir element; rewriting each term $x_\alpha z_\alpha$ in the sum as $z_\alpha x_\alpha + [x_\alpha z_\alpha] = z_\alpha x_\alpha + t_\alpha$, we see that c acts by the scalar $(\lambda + 2\rho)\cdot\lambda = ||\lambda+\rho||^2 - ||\rho||^2$ on the highest weight vector in M_λ, where 2ρ (denoted 2δ in the text) is the sum of the positive roots. Since the highest weight vector of M_λ generates M_λ as an L-module, c acts by this scalar on all of M_λ and by the same scalar on any irreducible subquotient of M_λ. But now we have already observed that the weights μ of M_λ all lie in the translate of the root lattice of L by λ; only finitely many vectors μ in this translate can have $\mu + \rho$ of the same square length as $\lambda + \rho$. The upshot is that *there are only finitely many simple subquotients of M_λ up to isomorphism.* Next time we will continue this argument to show that M_λ as finite length as a U-module and explore its structure in more detail.