Lecture 3-15

We wrap up the course by starting from a Chevalley basis of a complex simple Lie algebra, as we did to construct its split and compact real forms, but now moving in a different direction. We mentioned that if $x_\alpha, x_\beta, x_{\alpha+\beta}$ are root vectors in a Chevalley basis such that α, β, and $\alpha+\beta$ are all roots, then the constant $c_{\alpha\beta}$ appearing in the bracket $[x_\alpha x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$ is equal to $\pm(r+1)$ where r is the largest integer (possibly 0) for which $\beta - r\alpha$ is a root. If we now assume that $\beta+2\alpha, \ldots, \beta+q\alpha$ are all roots, then repeatedly bracketing x_β with x_α gives rise to the coefficients $\pm(r+1), \pm(r+1)(r+2), \ldots, \pm(r+1)\ldots(r+q)$, all of which remain integers upon division by $q!$, so we conclude that ad $x_\alpha^q/q!$ and exp ad x_α both leave the $\mathbb{Z}$-span $L' = L_{\mathbb{Z}}$ of a Chevalley basis of L invariant. Hence the group G generated by all exp ad cx_α as α runs over the roots also leaves L' invariant. Now if p is a prime, we can reduce everything modulo p: even though the resulting reduction L'_p of L' might not be semisimple, as noted last time, and even though some of the denominators arising in the definition of exp ad x_α might be equal to 0 mod p, the reduced group G_p still makes sense and still acts on the reduction of L'_p mod p. More generally, if F is any field of characteristic p, then we can set L'_F to be the algebra obtained from L'_p by extending scalars from $\mathbb{Z}_p$ to F, then the group G_F generated by exp ad cx_α as c runs over the elements of F and α runs over the roots is well defined and acts by automorphisms on L'_F. In a famous and fundamental paper of 1955, Chevalley showed that *the groups G_F are simple*, apart from a few exceptional cases for very small fields F. He thereby exhibited several families of finite simple groups; many of the groups in these families were previously known, but Chevalley's was the first unified treatment of them. The simplest example of a Chevalley group is obtained from $G_q = GL(n, \mathbb{F}_q)$, the group of $n \times n$ invertible matrices over the finite field $\mathbb{F}_q$ of order $q = p^n, p$ a prime. Here G_q itself is not simple, but if we pass first to the subgroup H_q of matrices in G_q with determinant 1 and then to the quotient P_q of H_q by its center, we do get a simple group for $q \geq 5$. There is a simple formula for the order of S; observing first that the first row of an $n \times n$ invertible matrix over $\mathbb{F}_q$ can be any nonzero vector in $\mathbb{F}_q$, then the second row can be any vector not a multiple of the first row, and so on, we see that that the order of G_q is $(q^n-1)(q^n-q)\ldots(q^n-q^{n-1} = q^{\binom{n}{2}}(q-1)\ldots(q^n-1)$. Dividing by $q-1$ (there are $q-1$ possible determinants of a matrix in G_q, each equally likely), and then finally by the order $g =$gcd$(n, q-1)$ of the cyclic center of H_q, we get the formula for the order of P_q, namely $q^{\binom{n}{2}}(q^2-1)\ldots(q^n-1)/g$. Chevalley showed that the orders of all finite Chevalley groups attached to $\mathbb{F}_q$ are given by a formula of much the same form, namely a power of q times a product of various differences $q^i - 1$, possibly divided by another constant equal to the order of the center of a certain finite group. The exponents i of the powers of q arising in the formula depend in a very interesting way on the root system of the Lie algebra giving rise to the group G_q; they are in fact called the exponents of this root system and have many applications not related to Chevalley groups.

Later Steinberg showed that whenever the Dynkin diagram of the Lie algebra L admits a nontrivial diagram automorphism, that automorphism can be folded into the construction of Chevalley groups to produce further families of finite simple groups, called *twisted*. In fact it turns out that in characteristic 2 there is a series of twisted groups called the *Suzuki* *groups* that do not have analogues in characteristic 0 but which eventually were seen to fit into this construction; later Ree constructed further analogues for type G_2 in characteristic 3. Then the alternating groups, the finite Chevalley groups, their analogues using diagram automorphisms in types A_n, D_n, and E_6 due to Steinberg, and the analogues due to Ree, Suzuki, and Tits and existing only in characteristics 2 and 3, account for all but finitely many finite simple groups of nonprime order. There are 26 *sporadic* finite simple groups as well, not lying in any infinite family of groups. The largest of these is the *Monster* group; it too turns out to have a Monster Lie algebra attached to it.