Lecture 3-13

We continue with real forms, constructing the noncompact real forms for each complex semisimple Lie algebra in turn. In type A, we have the set $\mathfrak{su}(p,q)$ of trace 0 skew-adjoint matrices with respect to the Hermitian form $(\cdot,\cdot)_{pq}$. This form is defined by $((v_1,\ldots,v_{i+j}),(w_1,\ldots,w_{i+j}))_{pq} = \sum_{k=1}^{p} v_k\bar{w}_k - \sum_{k=p+1}^{p+q} v_k\bar{w}_k$. We also have the split real form $\mathfrak{sl}(n,\mathbb{R})$. There is only one other real form, living only in even dimensions, and denoted $\mathfrak{su}^*(2n)$; it consists of all $n\times n$ matrices over the ring $\mathbb{H}$ of quaternions having trace a real combination of i,j,k.

In types B and D we once again have a family of (this time) symmetric bilinear forms on $\mathbb{R}^n$ parametrized by their signatures (p,q); here $(\cdot,\cdot)_{pq}$ is defined via the recipe $(v_1,\ldots,v_{p+1}),(w_1,\ldots,w_{p+1})_{pq} = \sum_{k=1}^{p} v_i w_i - \sum_{k=p+1}^{p+q} v_i w_i$. The set of skew-adjoint matrices with respect to this form is denoted $\mathfrak{so}(p,q)$; if $p=0$ or $q=0$ we get the compact form $\mathfrak{so}(p+q)$ defined earlier, while the split real form arises from the cases $p=q$ and $p=q\pm 1$. There is one further real form in type D_n denoted $\mathfrak{so}^*(2n)$; it consists of skew-adjoint $n\times n$ matrices over $\mathbb{H}$ with respect to the skew-Hermitian form $\sum_{i=1}^{n} v_i j\bar{w}_i$, where the j term in the middle refers to the quaternion with this label (not an index).

In type C we have the split form $\mathfrak{sp}(2n,\mathbb{R})$ and then the set $\mathfrak{sp}(p,q)$ skew-adjoint matrices over $\mathbb{H}$ with respect to the Hermitian form $\sum_{i=1}^{p} v_i\bar{w}_i - \sum_{k=p+1}^{p+q} v_i\bar{w}_i$.

To describe the real forms of the exceptional simple Lie algebras we need to say a bit more about the Cartan decomposition $L = K + P$ from last time; here L is a semisimple Lie algebra over $\mathbb{R}, K$ is a compact subalgebra of L such that the restriction of the Killing form κ on L to K is negative definite, and P is a K-submodule of L such that $[PP]\subset K$ an the restriction of κ to P is positive definite. We get an involution θ (automorphism of order 2) on L by decreeing that $\theta = 1$ on K and $\theta = -1$ on P; we call θ the *Cartan* involution. Then θ extends uniquely to a complex-linear involution on the complexification of L; it is equal to 1 on the complexification of K and -1 on the complexification of P. It turns out that conjugacy classes of automorphisms of order two of a complex Lie algebra classify its noncompact real forms up to isomorphism; in the exceptional case we can further label a real form of the complex Lie algebra of type X_N as $X_n(a)$ if $\dim P - \dim K = a$ (that is, no two nonisomorphic real forms of the same complex exceptional Lie algebra have the same value of a. We now run through the possibilities for X_N in turn.

The most interesting type is E_6; here we find four real forms, denoted $E_6(6), E_6(2)$, $E_6(-14)$, and $E_6(-26)$. The fixed subalgebras of the corresponding involution of the complex Lie algebra of this type have the respective types $C_4, A_5\times A_1, D_5\times\mathbb{C}$, and F_4. In type E_7 there are three noncompact real forms, denoted $E_7(7), E_7(-5)$, and $E_7(-25)$; here the fixed subalgebras have types $A_7, D_6 + A_1$, and $E_6\times\mathbb{C}$. In type E_8, there are just two noncompact real forms, denoted $E_8(8)$ and $E_8(-24)$; the fixed subalgebras have types D_8 and $E_7\times A_1$. In type F_4 we get the forms $F_4(4)$ and $F_4(-20)$, having fixed subalgebras of types $C_3\times A_1$ and B_4, respectively. Finally in type G_2, there is just one noncompact real form, labelled $G_2(2)$, and having fixed subalgebra of type $A_1\times A_1$. In all cases the form labelled $X_N(N)$ is the split one and whenever the label $\mathbb{C}$ appears in the type of a subalgebra it refers to the one-dimensional center of that subalgebra. In most cases we note that the rank of the subalgebra equals that of the complex algebra.

Where do these involutions and subalgebras come from? In fact they arise in a very simple way in most cases; the automorphism is a *diagonal automorphism*, acting as the identity on a Cartan subalgebra and as a scalar on every root space relative to it. More precisely, recall the extended Dynkin diagrams that we introduced earlier and used to classify Dynkin diagrams; they arise from ordinary (connected) Dynkin diagrams by adding one more vertex, representing the negative of the highest root, and joining it to the other vertices by the usual rules. We can label the vertices in this diagram, labelling the added vertex 1 and the other vertices by the coefficient of the corresponding simple root in the highest root (indeed, the extended Dynkin diagram is often understood to include these labels by definition). Now it turns out that, *except* in types A_n, D_n, and E_6, the only possibilities up to conjugacy for the involution in the exceptional cases either involve choosing just one vertex with the label 2, or else two vertices with the label 1. The corresponding fixed subalgebra then has diagram corresponding to the unchosen vertices, together with a copy of $\mathbb{C}$ if there are two chosen vertices with the label 1. The slight complication in the other types A_n, D_n, and E_6 arise from the existence of a *diagram* automorphism of order 2 in those types; here involutions involving that automorphism π come from a slight modification of the extended Dynkin diagram of the fixed point subalgebra of π(one where a vertex corresponding to the negative of a suitable multiple of a root rather than to the highest root is added).