Lecture 3-11

In the first week of class, when I gave basis vectors for the classical Lie algebras, you may already have noticed that the commutator of any two of them is an integral combination of the others. Now we are in a position to prove that any semisimple Lie algebra L over an algebraically closed field K of characteristic 0 admits a basis consisting of elements of a Cartan subalgebra and root vectors, such that the bracket of any two basis elements is a *rational* combination of basis elements. The slickest way to see this is to invoke the abstract construction by generators and relations of a semisimple Lie algebra over any field K having a specified root system Φ. The construction works over any field K of characteristic 0; the only reason that it can fail in general (and would not work over $\mathbb{Z}$) is that at one point it invokes the action of the Weyl group, which is defined by finite power series involving denominators. But in fact any semisimple complex Lie algebra L does admit a basis of this type such that the bracket of any two basis vectors is an integer combination of basis vectors. To prove this, let σ be the Chevalley automorphism of L mentioned earlier; it acts as -1 on a fixed Cartan subalgebra and sends any root space L_α to the negative root space $L_{-\alpha}$. If we choose any nonzero $x_\alpha \in L_\alpha$ for α positive, the bracket $[x_\alpha, -\sigma(x_\alpha]$ is a multiple of h_α, the element of H corresponding to $2\alpha/\alpha \cdot \alpha$. Multiplying x_α by $c \in \mathbb{C}$ multiplies $[x_\alpha, -\sigma(x_\alpha]$ by c^2. By the algebraic closure of $\mathbb{C}$, we can choose $x_\alpha \in L_\alpha$ for all positive α in such a way that $[x_\alpha, -\sigma(x_\alpha] = h_\alpha$ for $\alpha > 0$; replacing x_α by $-\sigma(x_\alpha)$ and using that $(-\sigma)^2 = 1, h_{-\alpha} = -h_\alpha$, we see that $[x_\alpha, -\sigma(x_\alpha)] = h_\alpha$ holds for all roots α. Having chosen $x_\alpha \in L_\alpha, -\sigma(x_\alpha \in L_{-\alpha}$ for all $\alpha > 0$, let $h_1, \ldots, h_r$ enumerate the h_β as β runs through the simple roots. Then every h_α is an integer combination of h_i, every h_i has every x_α as an eigenvector with integer eigenvalue, and if α, β, and $\alpha + \beta$ are all roots with $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$, then on applying σ we get that $c_{\alpha\beta} = -c_{-\alpha,-\beta}$. It is then shown in the text that these conditions imply that $c_{\alpha\beta} = \pm(r+1)$, where r is the largest positive integer such that $\beta - r\alpha$ is a root; in particular, all coefficients appearing are indeed integers. We call a basis of L consisting of elements of H and root vectors constructed in this way a *Chevalley basis*. Such a basis shows that a semisimple Lie algebra with a root space decomposition and given root system exists over any field (and even over any ring of characteristic 0). We call such a Lie algebra *split semisimple*; clearly it has properties quite analogous to those of such Lie algebras over algebraically closed fields.

Now take the basefield K to be $\mathbb{R}$ and let L be a semisimple *complex* Lie algebra with Chevalley basis $\{x_\alpha, h_i : \alpha \in \Phi, i = 1, \ldots, r\}$. Let L' be the *real* subalgebra spanned by $x_\alpha - x_{-\alpha}, i(x_\alpha + x_{-\alpha}, ih_j$, where α runs over the positive roots, the index j runs as above from 1 to r, and as usual i is a square root of -1. The property $c_{\alpha\beta} = c_{-\alpha,-\beta}$ noted above guarantees that L' is indeed closed under the bracket. Now L' is semisimple, but it does *not* have a root space decomposition; indeed, it turns out that all elements of L' are semisimple (but *not* diagonalizable; in fact all elements of L' have purely imaginary eigenvalues), but L' is not abelian. The Killing form κ is easily seen to restrict to a negative definite real bilinear form on L'; the adjoint group Int L', now generated by all exp ad x as x runs over all of L', not just the ad-nilpotent elements, preserves this form, so that Int L is naturally realized as a closed subgroup of $O(n, \mathbb{R})$, where n is the real dimension of L', equal to the complex dimension of L. Accordingly, we call *both* L' and its adjoint group *compact*, even though L' is obviously not compact as a topological space. The subalgebra L' is uniquely determined up to isomorphism by L; it is called the *compact real form* of the latter.

We can give a direct construction of L' for classical Lie algebras L, analogous to the definitions of those algebras we gave in the first week. Recall that the standard Hermitian form $(\cdot,\cdot)$ on $\mathbb{C}^n$ is defined via $((v_1,\ldots,v_n),(w_1,\ldots,w_n) = \sum v_i\bar{w}_i$; this form is complex-linear in the first variable but only conjugate-linear in the second. The set of all complex linear transformations X of trace 0 from $\mathbb{C}^n$ to itself that are skew-adjoint with respect to this form (so that $(Xv,w) = -(v,Xw)$ for all $v,w \in \mathbb{C}^n$) is a real (not complex) Lie algebra denotes $\mathfrak{su}(n,\mathbb{C})$; it realizes the compact real form of type A_{n-1}. In types B and D, we can similarly but more simply take L to consist of all linear transformation s from $\mathbb{R}^n$ to itself skew-adjoint with respect to the dot product, or even more simply of all skew-symmetric $n \times n$ real matrices. Finally, in type C, the intersection $\mathfrak{sp}(2n,\mathbb{C}) \cap \mathfrak{su}(2n,\mathbb{C})$ realizes the compact form we are looking for. This can also be described as the set of right linear transformations from *quaternionic* n-space $\mathbb{H}^n$ to itself that are skew-adjoint with respect to the Hermitian form $(\cdot,\cdot)$ (given by the above recipe, but now with $v_i, w_i \in \mathbb{H}$).

In general it turns out that there are up to isomorphism only finitely many *real forms* of a fixed complex semisimple Lie algebra L; that is, real subalgebras L' of L having the same dimension over $\mathbb{R}$ that L does over $\mathbb{C}$ and such that $\mathbb{C}L' = L$. Any such L' has a *Cartan decomposition* $K + P$ with K a compact subalgebra, P a K-submodule, and $[PP] \subset K$. Moreover the restriction of the Killing form κ on L' to K is negative definite and the restriction of κ to P is positive definite. We will say more about noncompact and nonsplit real forms later.