## Lecture 3-1

Last time we learned that any finite-dimensional irreducible module $V = V^\lambda$ over a semisimple Lie algebra $L$ is completely determined by its highest weight $\lambda$, which can be any dominant integral weight. The module $V^\lambda$ is the direct sum of its weight spaces $V^\lambda_\mu$; the weights $\mu$ occurring all lie below $\lambda$ and are such that $\mu$ occurs if and only if $w\mu$ does, for all $w$ in the Weyl group $W$ of $L$. The dimension $\dim V^\lambda_\mu$ of the $\mu$ weight space of $V^\lambda$ is called the *multiplicity* of $\mu$ in $V^\lambda$; we saw last time that this any two $W$-conjugate weights have the same multiplicity in $V^\lambda$ for any dominant integral $\lambda$. In particular, all $W$-conjugates $w\lambda$ of $\lambda$ itself have multiplicity one in $V^\lambda$. This observation is important because when we speak of the highest weight of a finite-dimensional representation we are implicitly making a choice of positive roots; no one choice is any better than another. Thus it must be that $V^\lambda$ has a unique highest weight with respect to any choice of positive roots. This is indeed the case; any two choices of positive roots are conjugate under $W$ and the corresponding highest weight spaces $W^\lambda_\lambda, W^\lambda_{w\lambda}$ are indeed both one-dimensional.

By contrast, suppose that $L$ admits a nontrivial automorphism $\pi$ of its Dynkin diagram, sending say the simple root $\alpha$ to another simple root $\beta \neq \alpha$. We have seen that $\pi$ corresponds to an automorphism $g_\pi$ of $L$, but how do we know (as claimed earlier) that $g_\pi$ is never inner in this situation? The answer is the unique irreducible module $V$ of highest weight $\lambda = \lambda_\alpha$, where $2\lambda_\alpha \cdot \alpha/(\alpha \cdot \alpha) = 1, 2\lambda \cdot \gamma = 0$ for simple $\gamma \neq \alpha$, is not isomorphic to the corresponding module $W$ with highest weight $\lambda' = \lambda_\beta$, since $\lambda \neq \lambda'$; but if $g_\pi$ were inner, it would act on $V$, sending its highest weight space to another highest weight space of weight $\lambda_\beta$. There is no such weight space in $V$, so $g_\pi$ is not inner. No inner automorphism of $L$ can permute the simple root spaces nontrivially.

Which dominant weights $\mu$ occur in $V^\lambda$ (with nonzero multiplicity)? We have already seen that if $\mu$ occurs, then it must lie below (or equal) $\lambda$ in the partial order; it turns out that this necessary condition is also sufficient. In fact, if any weight $\lambda'$ occurs in $V^\lambda$, then any dominant integral $\mu < \lambda'$ also occurs in $V^\lambda$. To prove this, write $\lambda - \mu = \nu = \sum_\alpha n_\alpha \alpha$, the sum taking place over simple roots $\alpha$; then we must have $\nu \cdot \alpha > 0$ for some $\alpha$, whence $\lambda' \cdot \alpha \geq \nu \cdot \alpha$ is strictly positive and $\lambda' - \alpha$ must occur in $V^\lambda$, by the representation theory of $\mathfrak{sl}(2)$. Continuing in this way, we see that $\mu$ also occurs in $V^\lambda$. In fact, the proof yields more: the representation theory of $\mathfrak{sl}(2)$ guarantees that the dimension of the $\lambda' - \alpha$ weight space is at least that of the $\lambda'$ weight space, whence ultimately the multiplicity of $\mu$ in $V^\lambda$ is at least that of $\lambda'$. Since weights conjugate under $W$ have the same multiplicity in $V^\lambda$, we see that *the necessary and sufficient condition for an arbitrary (integral) weight $\mu$ to occur in $V^\lambda$ is that $\mu$ and all of its $W$-conjugates either lie below $\lambda$ in the partial order or are equal to it.*

As you may have guessed, the highest root of any simple Lie algebra $L$ is also the highest weight of $L$ as an irreducible $L$-module. If $L$ is semisimple, then its highest weights as an $L$-module are just the highest roots of its simple components, each regarded as the 0 weight on the other simple components.

I have mentioned that there are finitely many but in general more than one complex Lie group with a fixed Lie algebra $L$ over $\mathbb{C}$; the different Lie groups $G$ with Lie algebra $L$ correspond to lattices lying between the root and weight lattices of $L$. It turns out that the action of $L$ on a finite-dimensional module lifts to $G$ if and only if the weights of the module lie in the lattice corresponding to $G$. Thus all finite-dimensional $\mathfrak{sl}(n,\mathbb{C})$ modules are also $SL(n,\mathbb{C})$ modules, but given the quotient $G_m$ of $SL(n,\mathbb{C})$ by the cyclic subgroup generated by the scalar matrix $e^{2\pi im/n}I$ for some $m|n$, we et a $G_m$-action on an $\mathfrak{sl}(n,\mathbb{C})$-module if and only if the coordinates of its weights all have denominators diving $m$. The situation is simpler for the other classical groups; in types $B$ weights either have only integers or only half-integers as coordinates and the simply connected group $\mathrm{Spin}(2n+1,\mathbb{C})$ or $\mathrm{Spin}(2n,\mathbb{C})$ acts in both cases but $SO(2n+1,\mathbb{C})$ and $SO(2n,\mathbb{C})$ does not. In type $C$ the coordinates of any weight are integers, but a weight lies in the root lattice if and only if the sum of its coordinates is even. The group $\mathrm{Sp}(2n,\mathbb{C})$ always acts on any finite-dimensional $\mathfrak{sp}(2n,\mathbb{C})$ module but its quotient $\mathrm{PSp}(2n,\mathbb{C}) = \mathrm{Sp}(2n,\mathbb{C})/\{I,-I\}$ acts on a finite-dimensional module if and only if its weights lie in the root lattice.