Lecture 2-8

We begin by constructing the Dynkin diagrams of the root systems we have seen (which it turns out is essentially all of them). In type A_{n-1} we get a single chain of $n-1$ vertices, each connected to its neighbors by a single edge. In type B_n we get a chain of n dots, with the rightmost edge a double one and having an arrow pointing to the rightmost vertex (and all other edges single). In type C_n we get the same diagram as in type B_n, except that the arrow points the other way. In type D_n we get the same diagram as for A_{n-1} except that there is a new vertex joined to the next to rightmost one; all edges are single. In type E_8 we get a single chain of seven vertices with an extra vertex joined to the third vertex in the chain; in types E_7 (resp. E_6) we get the same diagram with one vertex (resp. two vertices) removed from the long end of the chain. In type F_4 we get a chain of four vertices with the middle edge a double one and having an arrow pointing to the right. Finally, in type G_2 we get a pair of vertices jointed by a triple edge with an arrow pointing to the right. (In the non-crystallographic cases we get two vertices joined by an edge labelled m for $I_2(m)$, the root system consisting of the vertices and edge midpoints of a regular m-gon in the plane, together with their negatives. There are two remaining root systems H_3 and H_4; here H_3 consists of the edge midpoints of a regular icosahedron while H_4 consists of the centers of the faces of a 120-sided regular polytope in $\mathbb{R}^4$ called the hecatonicosahedroid. The Coxeter graph of H_3 is a chian of three vertices with the leftmost edge labelled 5; for H_4 it is a chain of four vetices with the leftmost edge labelled 5).

Now we are ready to classify all Dynkin diagrams of crystallographic root systems. We begin with the simple observation that a graph is a Dynkin diagram if and only if its connected components are Dynkin diagrams, for given any two roots systems Φ_1, Φ_2, living in $\mathbb{R}^n, \mathbb{R}^m$, respectively, we can write $\mathbb{R}^{n+m}$ as the orthogonal direct sum of $\mathbb{R}^n$ and $\mathbb{R}^m$ and accordingly realize the (disjoint orthogonal) union of the Φ_i as a new root system.

We have already seen the list of connected Dynkin diagrams, so it remain to see that the list is complete. What we do is classify all Coxeter graphs arising from bases of unit vectors in $\mathbb{R}^n$ such that the angle between any two of them is $\pi - \pi/m$ for some $m \in \{2, 3, 4, 6\}$; any simple subsystem of a root system give an example of such a basis, obtained by dividing every simple root by its length. We begin by exhibiting certain graphs that *cannot* occur. Given one of the root systems Φ described previously, one can observe in each case that there is a unique lowest root γ, in the sense that $\delta - \gamma$ is a positive combination of simple roots for every other root δ. We will work out what γ is in each case below, noting in all cases that γ has a nonpositive dot product with every simple root α. Now form the diagram attached to the old set of simple roots together with γ; this is called the *extended* or *affine* Coxeter graph. The set of vectors is a dependent set; more precisely there is a dependence relation among it in which all coefficients are positive integers. The extended graph cannot correspond to any subdiagram of the Dynkin diagram of any root system, for then the corresponding combination of simple roots would have square length 0. Moreover, any diagram obtained from an extended Coxeter graph by increasing the label(s) of any edge(s) also cannot correspond to any subdiagram of a Dynkin diagram, for the above combination of simple roots would still have nonpositive square length.

We now work out the root γ and construct the extended Coxeter graph for all crystallographic root systems X_N. In type A_{n-1}, we have $\gamma = e_n - e_1$; in type $B_n, \gamma = -e_1 - e_2$; in type $C_n, \gamma = -2e_1$; in type $D_n \gamma = -e_1 - e_2$. In type $E_8 \gamma = -e_8 - e_7$; in type $E_7 \gamma = e_7 - e_8$; in type $E_6 \gamma = (1/2)(-1, \dots, -1, 1, 1, -1)$. Finally, in type $F_4, \gamma = -e_1 - e_2$ and in type G_2 it is $(-2, 1, 1)$.

Constructing the extended Coxeter graphs, we get a closed cycle of n vertices in type A_{n-1}; in type B_n, the diagram of type B_n with an extra vertex joined to the second leftmost vertex; in type C_n, the extra vertex is joined by an arrow labelled 4 to the leftmost vertex; in type D_n, the extra vertex is joined to the second leftmost vertex. In type E_8, the extra vertex is joined to the long end; in type E_7, the extra vertex is joined to the short end; in type E_6, the extra vertex is joined to the topmost vertex (so overall we get three chains of vertices of length three, all sharing a common vertex). In types F_4 and $G-2$ the extra vertex is joined by an unlabelled edge to an end vertex.

Now it is not difficult to see that in fact any connected Dynkin diagram whose Coxeter graph does *not* contain any subgraph of the kind described in the last paragraph must be one of the diagrams in our list. (For the details see p. 37 of Humphreys's book "Reflection Groups and Coxeter Groups" cited previously). Hence this list is indeed complete.

The Coxeter and extended Coxeter graphs of type E are particularly interesting. All consist of three chains, say of lengths $p, q,$, and r, sharing a vertex in common. Looking at the triples (p, q, r) of positive integers that arise in this way, we get $(2, 3, 3)$ for E_6, $(2, 3, 4)$ for E_7, and $(2, 3, 5)$ for E_8. It turns out that these are exactly the triples (p, q, r) of positive integers such that $1/p + 1/q + 1/r > 1$. Similarly, the triples $((3, 3, 3), (2, 4, 4), (2, 3, 6))$ arising from the extended diagrams of type E are exactly those triples (p, q, r) such that $1/p + 1/q + 1/r = 1$. This sum of three reciprocals pops up frequently in mathematics and is intimately related to the behavior of triangles in Euclidean versus non-Euclidean geometry.