Lecture 2-6

We now head toward the classification of root systems, following the treatment in Humphreys's 1990 book "Reflection Groups and Coxeter Groups" rather than the text, and in particular considering non-crystallographic root systems as well. If Φ is a root system (not necessarily crystallographic), then we begin by setting up a total order on the ambient vector space R^n in which Φ lives, decreeing that a nonzero vector $(a_1, \ldots, a_n)$ is positive if and only if the first nonzero a_i is positive, so that the sum $\vec{v} + \vec{w}$ of any positive vectors $\vec{v}, \vec{w}$ is positive, as is any positive multiple $c\vec{v}$ of a positive vector v. We now have a notion of positive and negative roots in Φ; given any root α, exactly one of α and $-\alpha$ is positive, and if α, β are positive roots such that $\alpha + \beta$ is a root, then it is a positive root. Next, we consider all subsets S of the set Φ^+ of positive roots such that every $\alpha \in \Phi^+$ is a combination of vectors in S with nonnegative coefficients (for example, Φ^+ itself is one such subset). Let Δ be a minimal such subset, i.e. one not properly containing any other such subset. I claim that $(\alpha, \beta) \leq 0$ for any $\alpha, \beta \in \Delta$. Indeed, otherwise $s_\alpha \beta = \beta + c_\alpha \alpha$ is a root, for some $c_\alpha < 0$. If this root is positive it must be a positive combination of roots $\sum_\gamma x_\gamma \gamma$ of roots in Δ. If the coefficient x_β is less than one, then by subtracting $x_\beta \beta$ from both sides of $s_\alpha \beta = \sum_\gamma x_\gamma \gamma$ and dividing both sides by $1 - x_\beta$, we realize β as a positive combination of other roots in Δ, whence β can be deleted from Δ, a contradiction. If $x_\beta \geq 1$, then again subtracting $x_\beta \beta$ from both sides and dividing by $1 - x_\beta$ if $x_\beta \neq 1$, we realize β as a *negative* combination of elements in Δ and thus as a negative root (or 0 as a negative combination of roots in Δ), again a contradiction. Similarly, we get a contradiction if instead $s_\alpha \beta$ is a negative root.

My next claim is that the roots in Δ are linearly independent. Indeed, if we had a dependence relation $\sum_\alpha c_\alpha \alpha = 0$ as α runs over Δ, then first of all not all the nonzero coefficients c_α can be of the same sign, lest the combination be a positive vector. Transferring terms with negative coefficients over to the other side we get disjoint subsets Δ_1, Δ_2 of Δ and two nonzero equal combinations $\sum_{\alpha \in \Delta_1} c_\alpha \alpha = \sum_{\beta \in \Delta_2} c_\beta \beta$ for which the coefficients c_α, c_β are positive. But now the dot product of the right side $\sum c_\beta \beta$ with some β must be positive, while the dot product of α and β for every α appearing in the left side is nonpositive, another contradiction.

Thus we have a set Δ of linearly independent positive roots such that every positive root is a positive combination of roots in Δ and the angle between any two roots in Δ is obtuse or right. In the crystallographic case, we can strengthen these properties. First of all, given two nonproportional roots α, β with $(\alpha, \beta) > 0$, the positive integers $2(\alpha, beta)/(\alpha, \alpha), 2(\alpha\beta)/(\beta, \beta)$ have product at most 3 (by Cauchy-Schwarz), whence one of these integers is 1. Then one of $\alpha - \beta$ and $\beta - \alpha$, and thus both, are roots. Now, given Φ^+, let Δ consist of the indecomposable roots in Φ^+ (not expressible as a sum of two other roots in Φ^+). Then every root in Φ^+ is a positive *integral* combination of roots in Δ (so that every root in Φ is either a positive or a negative integral combination of roots in Δ). We call the roots in Δ *simple* and we call Δ a *simple subsystem*. The subsystem Δ is determined uniquely by the positive system Φ^+, consisting as it does exactly of the positive roots not expressible as positive combinations of two or more positive roots.

Now, given any two independent roots $\alpha, \beta \in \Phi$, the product $s_\alpha s_\beta$ of the reflections corresponding to α, β is a rotation in the plane spanned by α and β (through twice the angle between them). As both s_α, s_β lie in the finite Weyl group W, this rotation must be by a rational multiple of 2π. If $\alpha, \beta \in \Delta$, then since no root can be a combination of α and β with coefficients of different signs, we get the even stronger conclusion that the angle between α and β must be $\pi - \pi/m$ for some integer $m \geq 2$. In the crystallographic case we get the even stronger conclusion that the only possibilities for m are $2, 3, 4$, and 6, since $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$ must be a positive integer strictly less than 4. We can record all the angles between simple roots very neatly in a diagram, as follows: take a set of n points, one for every simple root α. If α and β are not orthogonal, join the corresponding vertices by edges, using one edge if the angle between α and β is $2\pi/3$, two edges if it is $3\pi/4$, and three edges if it is $5\pi/6$. In the latter two cases, add an arrow pointing to the shorter of α and β. Finally, if α, β are orthogonal, do not join the corresponding vertices by any edges. We obtain the *Dynkin diagram* D of the crystallographic root system Φ. (If Φ is not crystallographic, we replace D by the *Coxeter graph*, in which vertices α and β are joined by a an edge labelled m, if the angle between them is $\pi - \pi/m$ with $m > 2$; if $m = 2$, the edge is omitted, while if $m = 3$, the edge is present but the label 3 is omitted, by a standard convention).

Given any positive nonsimple root β, we know that β is a positive combination of simple roots α, whence there must be a simple root α with $(\beta, \alpha) > 0$. Reflecting β by α, we get another root, necessarily positive since it is a combination of simple roots involving at least one other than α with positive coefficient. Continuing in this way, we see that *given any positive root, some product of simple reflections (corresponding to simple roots) takes it to a simple root*. Since any conjugate $gs_\alpha g^{-1}$ of a reflection s_α by an orthogonal transformation g is the reflection $s_{g\alpha}$, we also see that all root reflections s_β lie in the subgroup of W generated by the simple reflections, whence *the Weyl group W is generated by the simple reflections*. Also, since every positive root is a product of simple reflections applied to a simple root, and the effect of these simple reflections can be computed once the angles between the simple roots are known, it follows that *the root system Φ can be recovered from its Dynkin diagram D*. Moreover, given the set Φ^+ of positive roots (with respect to some choice of positive vectors in $\mathbb{R}^n$) and $\alpha \in \Delta$, the corresponding simple subsystem, we see that the reflection s_α sends α to $-\alpha$ but any other positive root to another positive root (involving a simple root with a positive coefficient). Now if we have two distinct positive systems Φ^+ and Φ_1^+, with simple subsystems Δ, Δ_1, then there must be some α lying in Δ_1 such that $\alpha \notin \Phi^+$. Applying s_α to Φ_1^+, we get another positive system whose intersection with Φ^+ has one more element, namely $-\alpha$, than it had before. Continuing in this way, we see that any two positive systems, and thus any two simple subsystems, are conjugate under the group W, whence in particular the Dynkin diagram or Coxeter graph of a root system does not depend on the choice of simple roots.

In the classical cases, it is standard to decree that a nonzero $(a_1, \ldots, a_n) \in \mathbb{R}^n$ is positive if and only if the first nonzero a_i is positive. Then the simple roots in type A_{n-1} are the vectors $e_1 - e_2, \ldots, e_{n-2} - e_n$; in type B_n we get the same simple roots plus the root e_n; in type C_n we get the same simple roots plus the root $2e_n$; finally in type D_n we get the same simple roots plus the root $e_{n_1} + e_n$. In type E_n it is standard to make a different choice of positive vectors, decreeing that a nonzero $(a_1, \ldots, a_n)$ is positive if and only if the *last* nonzero a_i is positive. Then the simple roots in type E_8 turn out to be $1/2(1,-1,\ldots,-1,1), e_2-e_1, e_2+e_1, e_3-e_2, \ldots, e_7-e_6$. In type E_7 we get the same simple roots with the last one omitted; in type E_6 we get the same simple roots with the last two omitted. Finally, in types F_4 and G_2 we revert to our usual convention for positive vectors. The simple roots in type F_4 are e_2-e_3, e_3-e_4, e_4, and $1/2(1,-1,-1,-1)$ and those for G_2 are $(0,1,-1), (1,-2,1)$.