Lecture 2-27

Last time we observed that irreducible left modules of a Lie algebra L are also irreducible left modules of its enveloping algebra U; as such they may be identified with quotients U/M of U by a maximal left ideal M. If U is semisimple and its basefield K is algebraically closed of characteristic 0, let $L = H \oplus \oplus_{\alpha\in\Phi} L_\alpha$ be its root space decomposition (with H a Cartan subalgebra), so that Φ is a root system. Let Φ^+, Δ be a choice of positive subsystem and the corresponding choice of simple subsystem, respectively. If we let B be the sum of H and the positive root space L_α (for $\alpha \in \Phi^+$), then B is a solvable subalgebra of L. If now V is a finite-dimensional irreducible L-module, then Lie's Theorem guarantees that V has a one-dimensional submodule Kv, so that any $h \in H$ sends v to $\lambda(h)v$ for some $\lambda \in H^*$, while $L\alpha v = 0$ for $\alpha \in \Phi^+$. Now we can apply the PBW Theorem. Fix an ordered basis of L in which root vectors corresponding to negative roots come first, followed by a basis of H, followed by a basis of root vectors corresponding to positive roots. Realizing V as U/M as above, we see that M contains all basis vectors corresponding to positive roots, together with $h - \lambda(h)$ for all $h \in H$. Let I be the left ideal generated by just these elements. Then PBW says that U/I has as a basis all monomials in the negative root vectors, taking those vectors in fixed order (including the empty product, with the value 0); equivalently, *V is spanned by all such monomials in the negative root basis vectors applied to v.* But now since the toral subalgebra H acts diagonally on any such monomial with weight given by the corresponding sum of negative roots, it follows that *V has a weight space decomposition $\sum_\mu V_\mu$, where $V_\mu = \{w \in V : [hw] = \mu(h)w\}$, where all weights μ are obtained from λ by a subtracting a suitable sum of negative roots.* In particular, all weights μ lie below in λ in the partial order $<$ defined in the text (according to which $v < w$ for $v, w \in H^*$ if and only if $w - v$ is a sum of positive roots, or equivalently a sum of simple roots). We therefore call λ the *highest weight* of V in this situation.

Now as noted above V must take the form U/M where M is a maximal ideal containing the ideal I defined in the last paragraph. But any proper left ideal I' containing I is spanned by common eigenvectors for ad H, having weight equal to a sum of negatives of simple roots. Hence the weight 0 cannot occur in I', corresponding uniquely as it does to the monomial 1, which generates all of U, so the sum M of all proper left ideals I' containing I is again proper, being a sum of weight spaces corresponding to nonzero weights. This says that *M is the unique maximal left ideal containing I.* Hence any two irreducible left L-modules V, W with the same highest weight λ are isomorphic (even if they are infinite-dimensional).

Moreover, there is a simple necessary condition on the highest weight λ for the irreducible module V_λ to be finite-dimensional, which turns out to be sufficient. Indeed, for each simple root $\alpha \in \Delta$, let S_α be the simple subalgebra of L isomorphic to $\mathfrak{sl}(2)$ defined earlier; it is spanned by the one-dimensional root spaces $L_\alpha, L_{-\alpha}$ and their bracket $[L_\alpha L_{-\alpha}]$, a one-dimensional subalgebra of H. If $V - \lambda$ is finite-dimensional, it must be a sum of irreducible finite-dimensional S_α-modules; in any such module we know that any weight vector sent to 0 by $x_\alpha \in L_\alpha$ necessarily has h_α weight equal to a nonnegative integer. Hence $\lambda(h_\alpha) = 2(\lambda \cdot \alpha)/(\alpha \cdot \alpha) \in \mathbb{N}$, the nonnegative integers.

Now the converse holds as well: if λ satisfies this condition and if we have $n_i = 2(\lambda \cdot \alpha_i)/(\alpha_i \cdot \alpha_i) \in \mathbb{N}$, where $\Delta = \{\alpha_1, \dots, \alpha_r\}$, then let N^- be the subalgebra of L

spanned by the negative root vectors and let J be the ideal of $U(N^-)$ generated by $y_i^{n_i+1}$, where y_i is a nonzero root vector in $L_{-\alpha_i}$, then one checks that $I + J$ is a left ideal of U (since $x_j \in L_{\alpha_j}$ commutes with any power of y_i if $j \neq i$, while the commutator $[x_i y_i^{n_i+1}] \in I$ by a direct calculation following from the representation theory of $\mathfrak{sl}(2)$. Hence if $M = I + J$, then the quotient $V = U/M$ is spanned by weight vectors, exactly one of them having weight λ and the others having weight strictly below λ in the partial order $<$. All weight spaces V_μ with $\mu < \lambda$ have finite dimension, since there are only finitely many ways to write the difference $\mu - \lambda$ as a sum of negative roots. Then V is a sum of finite-dimensional S_i modules for each i, since the S_i-submodule of it generated by 1 is finite-dimensional by construction, and the sum of the finite-dimensional S_i-submodules of V is stable under left multiplication by L, so is all of V. Now we can imitate the argument in our earlier construction of a Lie algebra with root system Φ. Multiplication by $x_i \in L_{\alpha_i}$ acts locally nilpotently on V, as does multiplication by $y_i \in L_{-\alpha_i}$, so the product $(\exp\, x_i)(\exp\, -y_i)(\exp\, x_i)$ acts by a well-defined automorphism of V, acting on its weight spaces by the simple reflection s_i in the Weyl group W. Hence *weights μ, μ' conjugate under the Weyl group W of L are such that their weight spaces $V_\mu, V_{\mu'}$ have the same dimension.*

But now any *dominant* weight μ (having nonnegative dot product with all α_i) lying strictly below λ in the partial order must have smaller square length than λ, since if ν is a sum of simple roots than $\mu \cdot \nu$ and $\nu \cdot \nu$ are both nonnegative, so $(\mu + \nu) \cdot (\mu + \nu)$ is strictly larger than $\mu \cdot \mu$. We already know that any element of the weight lattice is W-conjugate to a dominant weight (keep conjugating by a simple reflection until the dot product with all simple roots is nonnegative), so only finitely many dominant weights μ can occur as weights of V (there are only finitely many vectors with square length bounded by a fixed constant in any lattice), and so only finitely many weights altogether occur in V, each with a finite-dimensional weight space. Hence V is finite-dimensional, as desired; we also see that any two W-conjugate weights μ, μ' in V have the same multiplicity (that is, weight spaces of the same dimension).