Lecture 2-25

Now that we have constructed the semisimple Lie algebra L over any algebraically closed field of characteristic 0 (and analogues of them over any field, or even any commutative ring, of characteristic 0), our remaining task this quarter is understand the finite-dimensional L-modules; by Weyl's Theorem this reduces to understanding their finite-dimensional irreducible modules. For this we need another general construction along the lines of the one we used to prove that for any root system Φ there is a Lie algebra having Φ as its root system. Let K be a field (arbitrary for now). Last time we defined the free Lie K-algebra on a set $x_1, \ldots, x_n$ of generators, as a Lie subalgebra of the tensor algebra T generated by the x_i (a free noncommutative algebra). Given a quotient F/I of F by a Lie ideal I, it is natural to hope that it embeds in the quotient T/J, where J is the two-sided ordinary ideal of T generated by I; then T/J would be an associative algebra containing the Lie algebra F/I in such way that the bracket operation in F/I is given by commutation in the associative algebra. Less abstractly, we might start with a finite-dimensional Lie algebra L over K and ask whether there is a larger associative algebra containing L such that commutation in it of elements of L matches the bracket operation in L. Of course, if L is semisimple and π is a faithful representation of L, regarded as in injective homomorphism from L to $\mathfrak{gl}(n, K)$ for some n, then $\mathfrak{gl}(n, K)$ is an example of such an algebra; but we want a single algebra large enough to capture all representations of L simultaneously.

A natural way to construct such an algebra U is start with the tensor algebra T on the underlying vector space of L and then mod out by the relations $xy - yx = [x, y]$ for $x, y \in L$; note that here we are starting with an associative algebra and moding out by a two-sided ideal, whereas last time we started with a free Lie algebra and moded out by a Lie algebra ideal. We call $U = U(L)$ the *universal enveloping algebra*, or just the *enveloping algebra*, of L. For example, if L is abelian, say with basis $x_1, \ldots, x_n$, then clearly U is just the polynomial ring over K (or free commutative K-algebra) in the x_i, regarded as independent variables. A famous theorem called the Poincaré-Birkhoff-Witt (or PBW) Theorem asserts that the basic features of U in this example are found in general. More precisely, given L and U defined as above, it asserts that *if $x_1, \ldots, x_n$ is an ordered basis of L, then the monomials $x_1^{m_1} \ldots x_n^{m_n}, m_i \geq 0$, form a basis of U.* It is fairly easy to see by induction on the length of a monomial that monomials of the above form span U; the basic idea of the proof is simply to note that if x_i, x_j are basis vectors of L and if $j > i$, then the product $x_j x_i$ of these basis elements in the "wrong" order is just $x_i x_j + [x_j x_i]$, while in turn $[x_j x_i]$ is a linear combination of the x_k. Iterating this calculation, we see that any m-fold product of x_i can be rewritten as a combination of monomials in the above form, each of total degree at most m. The hard part is to see that monomials in the above form are linearly independent in U. This is done by constructing a suitable representation of U on S, the polynomial ring over K on the basis $x_1, \ldots, x_n$ of L. The details are on pp. 93-4 of the text.

There is a more sophisticated way to look at the construction of U from L. Let $U_m = U_m(L)$ be the K-subspace of U spanned by products of at most m elements of L (or equivalently at most m generators x_i). Then U_{m-1} is clearly a subspace of U_m. Make the direct sum G of all the quotients $G_m = U_m/U_{m-1}$ for $m \geq 0$ (taking $U_{-1} = 0$ into a rings by decreeing that if u, v are cosets of U_{n-1} and U_n and U_{m-1} in U_m, respectively, and if u_m, v_n represent u, v, respectively, then uv is the coset of $u_m v_n$ in U_{m+n}/U_{m+n-1}. It is easy to check that the definition of uv does not depend on the choice of u_m, v_n. Then the images $\bar{x}_i$ in G of the generators x_i of U commute in G, so we get a map from the polynomial ring S to G. Then the PBW Theorem implies (and in fact is equivalent to) the assertion that *this map is an isomorphism*.

(As an interesting consequence we see that the noncommutative algebra U has no zero divisors; indeed, given nonzero $u, v \in U$, there are m, n such that the images of u, v in G_n, G_m, respectively, are nonzero; but then the image of uv in G_{n+m} is not zero, since polynomial rings over fields have no zero divisors, whence we cannot have $uv = 0$.)

By the construction of $U = U(L)$ we see that *(left) L-modules M may be naturally identified with left U-modules*, so that we can apply the methods of associative (but not commutative) rings to study L-modules. This will be our basic approach, but we start with the Lie-theoretic observation that if M is a finite-dimensional module over a semisimple Lie algebra L with Cartan subalgebra H, root system Φ, and Borel subalgebra B equal to the sum of H and all positive roots spaces L_α, then by the preservation of the Jordan form and Lie's Theorem M has a nonzero weight vector v which is sent to 0 by the derived subalgebra of B, so that there is $\lambda \in H^*$ with $hv = \lambda(h)v$ for all $h \in H$ but $L_\alpha v = 0$ for any positive root $\alpha \in \Phi$. If M is irreducible, then the weight λ turns out to be uniquely determined by M and is called its *highest weight*. We will parametrize finite-dimensional irreducible L-modules by their highest weights and determine which weights can occur as highest weights.