Lecture 2-22

One further conjugacy result, which I will not take the time to prove. Given a semisimple Lie algebra L over an algebraically closed field K of characteristic 0, let H be a maximal toral subalgebra and Φ the corresponding root system. Let Φ^+ be a positive subsystem of Φ. Then $B = H \oplus \oplus_{\alpha\in\Phi^+} L_\alpha$ is a subalgebra of L, whose derived subalgebra $\oplus_{\alpha\in\Phi^+} L_\alpha$ is easily seen to be nilpotent, so B is solvable; in an upcoming homework problem you will show that B is not properly contained in any other solvable subalgebra of L. We call B a *Borel* subalgebra of L; more precisely, any subalgebra B constructed as above is a called a *standard* Borel subalgebra of L (relative to H, Φ, and Φ^+). Then it turns out that any two Borel (that is, maximal solvable) subalgebras of L are conjugate under Int L. We will not need this result, but it plays a crucial role in the geometry of L, the quotient of Int L by the stabilizer of B being called the *flag variety* of Int L.

We now address the question of how one constructs a semisimple Lie algebra over a field K of characteristic 0 with a specified root system $\Phi$of rank n (lying in a Euclidean space $\mathbb{R}^n$ equipped with the usual dot product; we assume that the dot product takes rational values on $\Phi \times \Phi$). Fix a simple subsystem $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of Φ. We begin by recalling the *free noncommutative algebra* T on a set $x_1, \ldots, x_n$ of generators x_i; this is by definition spanned over K by all formal products of not necessarily distinct x_i, regarding all such products as linearly independent. The *free Lie algebra* F generated by the x_i is the Lie subalgebra of T generated by the x_i; it is spanned over K by all m-fold brackets of generators x_i (for all m), with the only dependence relations among these brackets being the ones arising from anticommuativity of the bracket operation and the Jacobi identity. If I is the ideal of F generated by $r_1, \ldots, r_m$, then the quotient F/I is said to be *generated* by the x_i with *relations* the r_i.

Applying this construction to our situation, we choose $3n$ generators, labelled h_i, x_i, and y_i for $1 \le i \le n$. The x_i and y_i correspond to the roots $\alpha_i, -\alpha_i$. As relations we start with $[h_i h_j] = 0$ for any i, j, while $[h_i x_j] = \frac{2\alpha_i\cdot\alpha_j}{\alpha_i\cdot\alpha_i} x_j, [h_i y_j] = \frac{-2\alpha_i\cdot\alpha_j}{\alpha_i\cdot\alpha_i} y_j$; we further have $[x_i y_i] = h_i$ for all i, while $[x_i y_j] = 0$ for $i \ne j$. We first examine the Lie algebra L_0 defined by these relations alone. This algebra is infinite-dimensional, but it has some useful structure. By constructing a suitable representation of it (see p. 97 of the text) one shows that L_0 is the vector space direct sum of the subalgebra Y of it generated by the y_i, the subalgebra X generated by the x_i, and the subalgebra H generated by the h_i; moreover, the x_i, y_i, and H_i are linearly independent in L_0. The subalgebra H is easy to understand, being just the span of the h_i (an abelian Lie algebra). The subalgebras Y and X are more complicated, but they have a graded structure: if the m-fold bracket of any set of x_i is given a grade equal to the corresponding sum of the α_i, then X is the vector space direct sum of the X_λ, as λ runs through the nonnegative integral combinations of the α_i, and $[X_\lambda X_\mu] \subset X_{\lambda+\mu}$; also L_λ consists exactly of the elements $x \in L_0$ such that $[hx] = \lambda(h)x$ for h in the span H of the h_i. A similar result holds for Y, where this algebra is graded by the nonpositive integral combinations of the α_i. In particular, the α_i weight space of L_0 is one-dimensional, being spanned by x_i, while the $-\alpha_i$ weight space is likewise one-dimensional, being spanned by y_i. The $k\alpha_i$ weight space of L_0 is 0 for $k \ne 0, 1, -1$, since no bracket of x's or y's can have that weight.

Now we impose finiteness conditions which have the effect of cutting L_0 down to exactly the size it must have to be a finite-dimensional semisimple algebra with root system Φ. We know that $\alpha_i - \alpha_j$ is not a root for $i \neq j$; reflecting by s_{α_j}, we see that $s_{\alpha_j}(\alpha_i) + \alpha_j = \alpha_i + k_{ij}\alpha_j$ is not a root, where $k = k_{ij} = \frac{-2\alpha_i \cdot \alpha_j}{(\alpha_j \cdot \alpha_j)} + 1$. We therefore impose the relation that $(\text{ad } x_i)^k x_j = 0$, and similarly for y_i, y_j. Initially we let I, J be the ideals of X, Y respectively, generated by these brackets; but then a calculation on p. 99 of the text shows that I, J are in fact both ideals of L_0. These ideals are graded elements, so the quotient of L_0 by the sum $I + J$ retains a graded structure. Then one checks that ad x_i, ad y_i are locally nilpotent in the quotient $L = L_0/(I + J)$ (given any element z of this quotient, some power of ad x_i and some power of ad y_i both send z to 0), so that (exp ad x_i)(exp ad $-y_i$)(exp ad x_i) is well-defined automorphism of L, acting on its H-root spaces by the reflection s_{α_i}. It follows that the Weyl group W corresponding to Φ acts on the root spaces of L sending each root space to another one of the same dimension. In particular, for any root λ that is W-conjugate to a simple root α_i, the dimension of the λ-root space of H in L is 1, while for any multiple $k\lambda$ of a root λ with $k \neq 0, 1, -1$ the dimension of the $k\lambda$-root space is 0. But now an exercise in the last HW set says that any $\lambda \in H^*$ is either a multiple of a root, or else some W-conjugate is a combination of simple roots in which some coefficients are positive and others negative. The λ-root space in the latter case has dimension 0, as no such weight can occur in L_0. The upshot is that the H-roots of L are exactly those in Φ and each root space is one-dimensional. But then L is indeed finite-dimensional and semisimple with root system Φ, as desired.