Lecture 2-20

Last time we saw that, given a semisimple Lie algebra L and an automorphism π of its root system Φ, and given a simple subsystem Δ of Φ, we can choose nonzero vectors x_α in all simple root spaces L_α, together with nonzero vectors $x_{\pi\alpha}$ in all root spaces $L_{\pi\alpha}$, and then there will be a unique automorphism g_π of L agreeing with π on the maximal toral subalgebra H and sending x_α to $x_{\pi\alpha}$ for all $\alpha \in \Delta$. Now in general, one difficulty in applying this result is that g_π will not send simple root spaces to simple root spaces, so it is hard to understand the product $g_\pi g_{\pi'}$ of two automorphisms $g_\pi, g_{\pi'}$ obtained in this way. There are two important exceptions. One occurs if π happens to be a diagram automorphisms, so that it does indeed permute the simple root spaces; we then see that the group A of all such diagram automorphisms may be naturally identified with a subgroup of the full automorphism group of L (but the nonidentity automorphisms in it are never inner, so that A is *not* a subgroup of Int L). The other exception occurs if π sends all root in Φ to their negatives. Then g_π will interchange the x_α and $-y_\alpha \in L_{-\alpha}$ for $\alpha \in \Delta$, while sending h_α to its negative; so the square of g_π is the identity automorphism. We call this g_π a *Chevalley automorphism*. It turns out to be inner if and only if π lies in the Weyl group of L.

In the special case where the automorphism π is an element of the Weyl group W, we can give a much more direct construction of the Lie algebra automorphism g_π. It suffices to do this in the case where $\pi = s_\alpha$, a single simple reflection, as we have seen that any element of W is a product of such reflections. Here we can just choose $x_\alpha \in L_\alpha, y_\alpha \in L_{-\alpha}$ in our usual way (so that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha y_\alpha]$ span a subalgebra S_α of L isomorphic to $\mathfrak{sl}(2)$) and then set $g_\pi =$(exp ad x_α)(exp ad $-y_\alpha$)(exp ad x_α). We have already observed that this automorphism acts on any S_α-module, interchanging its positive and negative weight spaces, so sending any root space L_β to $L_{s_\alpha\beta}$, as required. Note that g_π^2 acts by the scalar -1 on any even-dimensional irreducible S_α-module and the scalar 1 on any odd-dimensional such module, so even in this case we usually do not have $g_\pi^2 = 1$, as we would have to if W were a subgroup of Int L.

We now turn our attention to the possible dependence of the root system Φ of a semisimple Lie algebra L on the choice of maximal toral subalgebra H. To show that L alone determines Φ, it is enough to show that any two maximal toral subalgebras of L are conjugate under Int L. To do this we need to digress briefly to say a few words about the Zariski topology on K^n; the conjugacy result depends crucially on the algebraic closure of K.

The closed subsets of K^n in the Zariski topology are by definition the sets of common zeros of some collection S of polynomials in $k[x_1, \ldots, x_n]$. Hence any nonempty open set in this topology is the union of the set of *nonzeros* N_f of f for various nonzero polynomials f, and the intersection of any two such sets contains $N_f \cap N_g = N_{fg}$, so is nonempty. This is a fundamental difference between the Zariski and (say) the Euclidean topologies on (say) $\mathbb{C}^n$. Now there is a theory of morphisms (polynomial maps) from K^n to K^n which parallels the theory of smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$; any such map f has a differential df mapping the tangent space of K^n at any point x to the tangent space of K^n at $f(x)$. If this last map is an isomorphism at x, then the image of f contains a nonempty open subset of K^n containing $f(x)$, which is dense in K^n by our previous remarks. Now let L

more generally be any Lie algebra over K and H a maximal toral subalgebra. Decompose L under the action of H as $\oplus_{\alpha \in \Psi \subset H^*} L_\alpha$ as the sum of H-root spaces, by analogy with the root space decomposition of a semisimple Lie algebra, though here we do not pull H off of the direct sum, so that 0 is always one of our roots. Of course we cannot expect the subset Ψ of H^* to have any particular structure, but it is still a finite set. Fix a vector space complement H'' to H in $L_0 = C_L(H)$, the centralizer of H in L, and let H' consist of all $h \in H$ with $\alpha(h) \neq 0$ for any nonzero $\alpha \in \Psi$. Then the sum $\mathcal{H} = H' + H''$ is a Zariski-open subset of L_0; we call its elements *regular*. Let $b_1 \ldots, b_m$ be a basis for $\oplus_{\alpha \in \Psi, \alpha \neq 0} L_\alpha$ obtained as the union of bases for each L_α and for $k_1, \ldots, k_m \in K, h \in \mathcal{L}_0, x \in H'$ let $f(k_1, \ldots, k_m, h)(x) =$(exp ad $k_1 b_1) \ldots$(exp ad $k_m b_m)(x + h)$; this is a polynomial map and counting dimensions shows that its differential is surjective at any $x \in H'$. Now if H_1, H_2 are two maximal toral subalgebras, then by applying this map we see that some conjugate of a regular element of H_1 is a regular element of H_2, Passing to the set of semisimple elements in the centralizers of these regular elements and using that the only semisimple elements centralizing H_i lie in H_i, we get that H_2 is conjugate to H_1, as desired. In particular, any two maximal toral subalgebras of a semisimple Lie algebra L have the same dimension, called the *rank* of L.

At this point I should mention that maximal toral subalgebras of a semisimple Lie algebra L are usually called *Cartan subalgebras* in the literature. They are actually defined as nilpotent subalgebras H of L equal to their own normalizers, so that if $[xH] \subset H$ for some $x \in L$, then $x \in H$; but this definition turns out to be equivalent to that of maximal toral subalgebra for semisimple L. In general, any Lie algebra L over an algebraically closed field has Cartan subalgebras in this sense and any two of them are conjugate under Int L.