Lecture 2-15

One last remark (for now) before we finally return to Lie algebras. Given a root system Φ, what is its automorphism group (that is, the group of orthogonal transformations on the ambient Euclidean space permuting the roots of Φ)? Any such automorphism g sends a positive subsystem Φ^+ of Φ to another such subsystem; then there is an element of w of the Weyl group W sending the second subsystem to the first. The composition wg is then an automorphism of Φ preserving a particular subsystem of positive roots, and hence also the corresponding simple subsystem. It therefore acts on the Dynkin diagram of Φ by an automorphism. Nontrivial such automorphisms exist (for connected Dynkin diagrams) only in the cases A_n, D_n, and E_6; in most of these cases there just one such automorphism, of order 2, but in the very interesting case of D_4 there are three "outer" simple roots that can be permuted arbitrarily by diagram automorphisms, so the group of diagram automorphisms is S_3, the symmetric group on three letters. This group turns out to act on the corresponding Lie algebra $\mathfrak{so}(8)$ by a very interesting set of automomorphisms; this phenomenon is called *triality* and can be used to give an explicit construction of the exceptional Lie algebras starting with the classical algebra $\mathfrak{so}(8)$ and using some of its irreducible modules.

Now at last we take up Lie algebras again. The first point is to see that any Lie algebra admitting a root space decomposition satisfying the axioms of a root system and one other mild condition is semisimple. More precisely, *suppose that a Lie algebra L admits a grading $H \oplus \oplus_{\alpha\in\Phi} L_\alpha$, where H is an abelian subalgebra acting diagonally on the space L_α by $\alpha \in H^*$, the space L_α are one-dimensional, the set Φ is a root system in H^*, and $[L_\alpha L_\beta] = L_{\alpha+\beta}$ whenever α, β are two roots such that $\alpha+\beta \in \Phi$. Then L is semisimple.* To prove this, we can reduce to the case where Φ has a connected Dynkin diagram, as clearly L is the direct sum of the subalgebras L' corresponding to the connected components of Φ. Now any nonzero ideal I of L, by a Vandermonde determinant argument, contains a root vector lying in some L_α; by choosing an appropriate set of simple roots, we may assume that α is simple, so that it corresponds to a vertex in the Dynkin diagram. Moving back and forth between root spaces and H via repeated brackets, and moving from one vertex in this diagram to its neighbors, we get the root space L_α in I for all simple roots α, as well as $L_| - \alpha \subset I$. But now we saw last time that any positive root can be written as a sum of simple roots in such a way that every partial sum is also a root; as a consequence, the root spaces L_α for α simple generate all roots space L_β for β positive, while similarly the root spaces $L_{-\alpha}$ generated $L_{-\beta}$. The upshot is that all positive negative root spaces of L lie in I, as does H, so finally $I = L$ is simple, as claimed.

Now suppose that L, L' are two simple Lie algebras (over the same field) with the same root system. More precisely, *suppose that there is a linear isomorphism π from a maximal toral sualgebra H of L to a maximal toral subalgebra H' of L' inducing an isomorphism between the root systems Φ, Φ' of L, L', respectively. Choose simple subsystems Δ, Δ' of Φ, Φ' so that this isomorphism maps Δ onto Δ' and choose nonzero vectors $x_\alpha, x'_{\alpha'}$ in $L_\alpha, L'_{\alpha'}$, respectively. Then there is a unique Lie algebra isomorphism from L to L' extending π and sending x_α to $x-_{\alpha'}$ for all $\alpha \in \Delta$.*

To prove this, first choose nonzero $y_\alpha \in L_{-\alpha}, y'_{\alpha'} \in L'_{-\alpha'}$ so that $h_\alpha = [x_\alpha y_\alpha], h'_{\alpha'} = [x_{\alpha'} y_{\alpha'}]$ combine with x_α, y_α (respectively $x_{\alpha'}, y_{\alpha'}$) to span subalgebras $S_\alpha, S_{\alpha'}$ of L, L'

both isomorphic to $\mathfrak{sl}(2)$. Then π necessarily sends h_α to $h_{\alpha'}$, so the extension of π necessarily sends y_α to $y_{\alpha'}$. Since the x_α, y_α together generate L we see at once that the extension of π is unique, if it exists. To show that the extension exists, look at the subalgebra D of the direct sum $L \oplus L'$ generated by all $(x_\alpha, x'_{\alpha'}), (y_\alpha, y'_{\alpha'}, (h_\alpha, h'_{\alpha'})$; if the extension exists, this must be a proper subalgebra isomorphic to both L and L'. Let β, β' be the highest roots of L, L' and choose nonzero $x_\beta \in L_\beta, x'_{\beta'} \in L'_{\beta'}$. Let $M \subset L \oplus L'$ be spanned by $(x, x') = (x_\beta, x'_{\beta'}$ and all vectors (v, v') obtained from (x, x') by applying a product P of ad y_α's to x and the corresponding product P' of ad $y'_{\alpha'}$'s to x', as α ranges over Δ. Since the difference of two simple roots in Φ of Φ' is never a root, one checks inductively that M is stable under the actions of both $(h_\alpha, h'_{\alpha'})$ and $(x_\alpha, x'_{\alpha'})$ for any $\alpha \in \Delta$, so M is a D-submodule. It is a proper subspace of $L \oplus L'$ since its intersection with the (β, β') root space of this direct sum is only one-dimensional. But then D must be a proper subalgebra of $L \oplus L'$, as otherwise M would be an ideal of $L \oplus L'$ and thus equal to L or L' (both of which are impossible). Once we know that D is a proper subalgebra, we are done: simplicity of L and L' guarantee that the kernel of either coordinate projection when restricted to D is trivial, whence D is indeed isomorphic to both L and L', as desired.

The theorem extends easily to semisimple Lie algebras L, by simply applying it to each simple component of L. It shows that two semisimple Lie algebras with the same root system are isomorphic, but it also shows more: given any automorphism π of the root system Φ of a semisimple Lie algebra L and any choice of nonzero root vectors $x_\alpha, x_{\pi\alpha}$ from the α and $\pi\alpha$ root spaces of L, there is a unique automorphism of L acting on a maximal toral subalgebra H by π and sending x_α to $x_{\pi\alpha}$ for every root α lying in a simple subsystem of Φ. We will work out some consequences of this last fact on Wednesday.