## Lecture 2-13

Last time we classified the connected Dynkin diagrams; we conclude with some further remarks about the lowest root $\gamma$ of a root system $\Phi$ with connected diagram arising in the classification. The negative $\beta = -\gamma$ of $\gamma$ is naturally enough called the *highest root*. We can easily work out the coefficient of every simple root $\alpha$ in $\beta$, as follows. Start with the extended Dynkin diagram of $\Phi$ (not its Coxeter graph; this time we include the arrows). Attach the label 1 to the added vertex. Now work out labels attached to the other vertices inductively, by the rule that the labels attached to vertices adjacent to a given one $v$ must add to twice the label of $v$ itself, counting the label of an adjacent vertex doubly if it lies at the long end of a double arrow, triply if it lies at the long end of a triple arrow, and singly otherwise. (The rule is justified by the observation we made in the course of the classification that the labels correspond to a combination of roots whose square length is 0). For example, all labels in type $A_{n-1}$ are 1; the labels in type $B_n$ are 1 for the two leftmost vertices and 2 for the other vertices, and so on. In type $E$, the most interesting case, the labels are, starting at the central vertex (of degree 3) and radiating outward along each chain, $3,2,1,3,2,1,3,2,1$ in type $E_6$; $4,3,2,1,4,3,2,1,4,2$ in type $E_7$; $6,5,4,3,2,1,6,4,2,6,3$ in type $E_8$.

Now there are two natural lattices attached to any (crystallographic) root system $\Phi$, one consisting of the integral span of the roots and naturally called the *root lattice*, the other consisting of all $\lambda$ in the ambient vector space $\mathbb{R}^n$ such that $2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. The second lattice is called the *weight lattice*; it is traditionally denoted by $P$ and the root lattice by $R$. As $P, R$ are free abelian groups of rank $n$, the quotient $P/R$ is a finite abelian group called the *fundamental group*. It is important because it turns out that complex semisimple Lie *groups* with a fixed semisimple Lie algebra $L$ are parametrized by subgroups of this group (or equivalently by subgroups of $P$ containing $R$). We have already seen the adjoint group of $L$, which corresponds to the trivial subgroup; the one corresponding to the fundamental group itself is the simply connected universal cover of this group. In type $A_{n-1}$, for example, this simply connected covering group is $SL(n,\mathbb{C})$, the group of $n \times n$ complex matrices of determinant one. Other groups with the same Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ are quotients of $SL(n,\mathbb{C})$ by a a subgroup of its center, which is a cyclic group of order $n$; correspondingly, the fundamental group in type $A_{n-1}$ is also cyclic of order $n$. In types $B$ and $D$, the simply connected covering group is called $\mathrm{Spin}(n,\mathbb{C})$; it is a double cover of $SO(n,\mathbb{C})$. The fundamental group has order two in type $B$ and order 4 in type $D$; it is cyclic in type $D_n$ if $n$ is odd but the product of two cyclic groups in type $D_n$ if $n$ is even.

What does all of this have to do with the highest root? It turns out that the cosets of $R$ in $P$ are uniquely represented by *minimal dominant integral weights* $\lambda$, which by definition are such that $2(\lambda,\alpha)/(\alpha,\alpha)$ is a nonnegative integer if $\alpha$ is simple and is at most one if $\alpha$ is positive (see Exercise 13.13 in the text). Now if every root $\alpha \in \Phi$ is replaced by its *coroot* $2\alpha/(\alpha,\alpha)$ the result is a root system $\Phi'$ which is said to be *dual* to $\Phi$; the coroots of a simple subsystem for $\Phi$ provide a simple subsystem for $\Phi'$. Then a nonzero dominant integral weight $\lambda$ (satisfying the first condition in the definition of minimal dominant integral weight) is minimal if and only if it is fundamental in the sense that $2(\lambda,\alpha)/(\alpha,\alpha)$ equals 1 for some simple root $\alpha$ and 0 for the other simple roots, *and* the coroot of $\alpha$ appears with coefficient one in the highest coroot. Hence the order of the fundamental group is one more than the number of ones appearing when the highest coroot is written as a combination of simple coroots (one more because the 0 coset corresponds to the 0 minimal dominant integral weight). This number is $n$ in type $A_{n-1}, 2$ in types $B$ and $C, 4$ in type $D, 3$ in type $E_6, 2$ in $E_7$, and 1 in the remaining types.

Before leaving the study of root systems we want to give a very nice formula for the order of any Weyl group $W$. This formula results from an application of the Orbit Formula for group actions coupled with Lemma 10.3B in the text. Looking at the action of $W$ on the highest root, we find that its orbit consists of all roots with the same length. An element of $W$ fixes the highest root if and only if it is a product of reflections fixing this root; consequently, the stabilizer of the highest root is the Weyl group corresponding to the simple roots orthogonal to the highest root. Applying this formula in type $E_6$, we find that the orbit of the highest root has size 72 (the number of all roots), while the stabilizer of the highest root is a Weyl group of type $A_5$ and so has order 6!. Multiplying, we get the order of $W$: it is $2^7\, 3^4\, 5$. Similarly, the order of $W$ in type $E_7$ is 126 times the order of $W$ in type $D_6$, or $2^{10}\, 3^4\, 5\, 7$. The order of $W$ in type $E_8$ is $2^{14}\, 3^5\, 5^2\, 7$. Finally, the order of $W$ in type $F_4$ is $2^7\, 9 = 1152$.