Lecture 2-1

Picking up from last time, let L be a semisimple Lie algebra, H a maximal toral subalgebra, and Φ the corresponding root system, so that the root space decomposition of L is $H \oplus \oplus_{\alpha\in\Phi} L_\alpha$. Last time we saw that all roots in Φ lie in a $\mathbb{Q}$-vector space of dimension $\dim_K H$. If $\alpha, \beta \in \Phi$, then $(\alpha, \beta) = \kappa(t_\alpha, t_\beta) = \sum_{\gamma\in\Phi}(\gamma, \alpha)(\gamma, \beta)$, by definition of κ. In particular, $(\beta, \beta) = \sum_{\gamma\in\Phi}(\gamma, \beta)^2$. Dividing by (β, β) and using that all ratios $2(\gamma\beta)/(\beta, \beta)$ are integers, we get that $(\beta, \beta) \in \mathbb{Q}$, whence $(\alpha, \beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$. Then finally for all nonzero $\lambda \in E_{\mathbb{Q}}$, the rational span of Φ, we have that $(\lambda, \lambda) = \sum_{\alpha\in\Phi}(\lambda, \alpha)^2$ is a sum of squares of rational numbers and so is positive. Replacing $E_{\mathbb{Q}}$ by $E_{\mathbb{R}}$, the real vector space (formally) spanned by Φ, we see that $(\cdot, \cdot)$ induces a positive definite symmetric real-valued bilinear form on $E_{\mathbb{R}}$, which we may identify with the usual dot product.

At this point we can forget about the Killing form and the basefield K. All that matters is that we now have a finite collection Φ of vectors in $\mathbb{R}^n$ for some n that span $\mathbb{R}^n$ and are such that if $\alpha \in \Phi$, then the only multiples of α lying in Φ are $\pm\alpha$, and if $\alpha, \beta \in \Phi$, then $\beta - 2(\beta, \alpha)/(\alpha, \alpha)\alpha \in \Phi$ and $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$. We call any collection Φ of vectors in some $\mathbb{R}^n$ satisfying these properties a *(crystallographic) root system*.

The expression $2(\beta, \alpha)/(\alpha, \alpha)$ arises in linear algebra over $\mathbb{R}^n$ in the formula for a reflection: if $\alpha \in \mathbb{R}^n$ is nonzero, then the unique linear transformation sending α to $-\alpha$ and any vector γ orthogonal to α to itself. (The hyperplane of vectors orthogonal to α is called the *reflecting hyperplane* and this reflection is denoted s_α; note that $s_{k\alpha} = s_\alpha$ for all nonzero k). The subgroup W of the orthogonal group $O(n, \mathbb{R})$ generated by all reflections s_α as α runs over Φ is called the *Weyl group* of Φ if Φ is crystallographic and its *Coxeter group* in general; it is finite since it acts faithfully on the finite set Φ. The axioms of a crystallographic root system can be weakened: if we do not require that $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$, then we call Φ just a root system (though often in the literature root systems are understood to be crystallographic). The crystallographic property amounts to requiring that the *root lattice*, i.e. the $\mathbb{Z}$-span of the roots, be preserved by the reflections s_α. We will classify crystallographic root systems next week and briefly indicate what the non-crystallographic root systems are as well.

For now we return to the classical (linear) Lie algebras discussed in the first week and compute their root systems. In all cases, we saw that the diagonal matrices in the Lie algebra form an abelian subalgebra H of semisimple elements whose centralizer is just H (as one sees by looking at the bases we constructed). Denoting the unit coordinate vectors in $\mathbb{R}^n$ as usual by $e_1, \ldots, e_n$ and the dot product by $(\cdot, \cdot)$, we note first that the reflection $s_{e_i - e_j}$ attached the difference of the ith and jth unit coordinate vectors acts on a vector by flipping its ith and j coordinates; similarly, the reflection s_{e_i} acts by changing the sign of the ith coordinate, and the reflection $s_{e_i+e_j}$ acts by flipping the ith and jth coordinates and changing their signs.

We saw then the first week that the root system in type A_{n-1} consists of all vectors $e_i - e_j$ as the distinct indices i, j run from 1 to n; notice again that the vector space they span has dimension only $n - 1$, thus accounting for the $n - 1$ subscript. The root system in type B_n consists of all vectors $e_i - e_j$ as above together with all vectors $e_i + e_j$ and e_i; now they span all of $\mathbb{R}^n$. The root system in type C_n is the same as in type B_n, except that the roots e_i are replaced by $2e_i$. Finally, the root system in type D_n is the same as in type B_n, omitting the roots e_i. Now it is is very easy to see that the axioms of a root system are indeed satisfied in all of these cases. We can give a uniform description of the systems of types B,and D as follows: in both cases we take all vectors of the appropriate square length or lengths in the lattice $\mathbb{Z}^n$ sitting inside the $\mathbb{R}^n$, using square length 2 for type D_n and lengths 1 or 2 for type B_n. For A_{n-1} we do the same as for D_n, replacing R^n by the hyperplane consisting of all vectors whose coordinates sum to 0 and $\mathbb{Z}^n$ by its intersection with this hyperplane. In type C_n, we just start with the root system for type B_n and replace the roots e_i by $2e_i$, as mentioned above. Note that in this case roots of square length 4 occur (but there are vectors of square length 4 in $\mathbb{Z}^n$ which do not lie in Φ).

In all the classical cases the Weyl group W is a familiar symmetry group: in type A_{n-1} it is the group S_n of permutations of n coordinates, of order $n!$; in types B_n and C_n it consists of all permutations and sign changes of n coordinates and has order $2^n n!$; and finally in type D_n it consists of all permutations and *evenly many* sign changes of n coordinates and has order $2^{n-1}n!$

In the exceptional cases we again take vectors of one or two fixed square lengths in a lattice. Our first new type is called E_8; here we take the vectors of square length two in the lattice in $\mathbb{R}^8$ spanned by all $e_i - e_j, e_i + e_j$, and $(1/2)(1, \ldots, 1)$: the lattice consists of all vectors $(a_1, \ldots, a_8)$ where the coordinates a_i are either all integers or all half-integers and in addition $\sum a_i$ is an even integer. The vectors in Φ consist of all sums $e_i + e_j$, differences $e_i - e_j$, and vectors $(\pm 1/2, \ldots, \pm 1/2)$, where evenly many 1/2s occur. There are 240 such roots. There are two other types with the label E, namely E_7 and E_6; for E_7 we take all roots in E_8 orthogonal to $e_7 + e_8$, while for E_6 we take all roots in E_8 orthogonal to both $e_7 + e_8$ and $e_6 + e_8$. There are 126 roots in type E_7 and 72 roots in type E_6. Finally, we have the root systems F_4 and G_2; for F_4 we take all vectors of square length one or two in the lattice in $\mathbb{R}^4$ spanned by the e_i and $(1/2, /1/2, 1/2, 1/2)$, consisting of all vectors of the form $\pm e_i, \pm e_i \pm e_j$ for $i \neq j$ and all vectors of the form $(\pm 1/2, \ldots, \pm 1/2)$, where the signs may be chosen independently. There are 48 roots. In G_2 we take our ambient vector space to be the hyperplane in $\mathbb{R}^3$ orthogonal to $(1, 1, 1)$ and the lattice to be spanned by $e_1 - e_2, e_2, e_3$. Take all vectors of square length 2 or 6 in this lattice, thereby obtaining all differences $e_i - e_j$ and $\pm(2, -1 - 1), \pm(-1, 2, -1)$, and $\pm(-1, -1, 2)$. There are 12 such roots.