Lecture 1-9

Continuing from last time, we now study the classical algebras (those belonging to one of the series A_n, B_n, C_n, or D_n), in more detail. Recall that the *matrix unit* e_{ij} has 1 as its ij-entry while all other entries are 0. There is a simple rule for commutating two such units: $[e_{ij}e_{k\ell}] = \delta_{jk}e_{i\ell} - \delta_{\ell}ie_{kj}$, where δ_{ij} is the Kronecker delta. In all cases the diagonal matrices will play a crucial role, since if d is diagonal than ad d will continue to act diagonalizably on the algebra containing d.

The simplest case is $L = \mathfrak{sl}(n)$ (type A_{n-1}). Here we have $[e_{ii}e_{ij}] = e_{ij}$, so that if $d = \sum_i d_ie_{ii}$, then $[de_{ij}] = (E_i - E_j)(d)e_{ij}$, where E_i is the linear function on the space of diagonal matrices sending one of them to its ii-entry. Here we must have $\sum_i d_i = 0$, since we are only looking at matrices with trace 0. Thus the dimension of the subspace D of diagonal matrices has dimension $n-1$ rather than n; this is why we say that L is of type A_{n-1} rather than A_n. This subspace D is a subalgebra of L, in fact an *abelian* one where all brackets are 0. (This will be true of the subspace D in all the other classical algebras). Here L is spanned by D together with all matrix units e_{ij} with $i \neq j$. The dimension of L is thus $n^2 - 1$.

In all the other families of examples, there is a matrix M such that the Lie algebra consists of all matrices X that are skew-adjoint with respect to the from $(cdot, \cdot)$ given by $(v, w) = v^tMw$, where the superscript t as usual denotes transpose, so that v^t is a row vector while w is a column one. Skew-adjointness with respect to this form translates to the condition on X that $MX = -X^tM$. We now consider each of the cases in turn.

Starting with $L = \mathfrak{sp}(2n)$ (type C_n), the condition $MX = -X^tM$ says that the upper left $n \times n$ block m of X should be the negative transpose $-q^T$ of the lower right block q, while the upper right and lower left blocks n, p should equal their own transposes. Hence a typical diagonal matrix d in L takes the form $d = \sum_i d_ie_{ii}$, where $d_{n+i} = -d_i$ for $1 \leq i \leq n$. Letting E_i as above be the linear function sending $\sum_i d_ie_{ii}$ to d_i (for $1 \leq i \leq n$ only), we find that M is spanned by D together with the differences $\ell_{ij} = e_{ij} - e_{n+1,n+j}$ (for $1 \leq i, j \leq n, i \neq j$), the sums $\ell'_{ij} = e_{i,n+j} + e_{j,n+i}$ (for $1 \leq i < j \leq n$), their transposes $\ell''_{ij} = e_{n+j,i} + e_{n+i,j}$, and the units $\ell_i = e_{i,n+i}$ and their transposes $\ell'_i = e_{n+i,i}$. Then $[d\ell_{ij}]$ equals $(E_i + E_j)(d)\ell_{ij}$ if $i < j$ and $(-E_i - E_j)(d)\ell_{ij}$ if $j < i$, while $[d\ell'_{ij}] = (E_i - E_j)(d)\ell'_{ij}, [d\ell''_{ij} = (E_j - E_i)\ell''_{ij}, [d\ell_i] = 2E_i(d)\ell_i, [d\ell'_i = -2E_i(d)\ell'_i$. The dimension of L is $n + n^2 - n + 2(n + (1/2)(n^2 - n)) = 2n^2 + n$; the dimension of D is n.

Now consider $L = \mathfrak{so}(2n+1)$ (type B_n). Now the condition $MX = -X^tM$ says that X has 0 as its11-entry; the remaining blocks of entries b_1, b_2 in its first row are the respective negative transposes of c_2, c_1, the remaining blocks of entries in its first column. The remainder of X consists of four blocks m, n, p, q as in the previous case, but this time with $q = -m^T, n^T = -n, p^T = -p$. Here D consists of all sums $\sum_i a_ie_{ii}$ with $d_1 = 0, d_{n+i} = -d_i$ for $2 \leq i \leq n+1$ and L is spanned by D together with all differences $\ell_{ij} = e_{i+1,j+1} - e_{n+j+1,n+i+1}$ (for $1 \leq i, j \leq n, i \neq j$), all differences $\ell'_{ij} = e_{i+1,n+j+1} - e_{j+1,n+i+1}$ (for $1 \leq i < j \leq n$), all differences $\ell''_{ij} = e_{n+i+1,j+1} - e_{n+j+1,i+1}$ (for $1 \leq j < i \leq n$), and all differences $\ell_i = e_{1,n+1+i} - e_{i+1,1}, \ell'_i = e_{1,i+1} - e_{n+1+1,1}$ (for $1 \leq i \leq n$). We have $[d, \ell_{ij}] = (\pm(E_i + E_j)(d)\ell_{ij}$ (according as $i < j$ or $j < i$, as before), while we have $[d, \ell'_{ij}] = (E_i - E_j)(d)\ell'_{ij}, [d\ell''_{ij}] = (E_i - E_j)(d)\ell''_{ij}, [d\ell_i] = E_i(d)\ell_i, [d\ell'_i] = -E_i(d)\ell'_i$, where $E_i(d) = d_{i+1}$ for $1 \leq i \leq n$. Here the dimension of D is again n and the dimension of L is again $2n^2 + n$. Notice that we get the same linear functions arising on D as in type C_n, except that $\pm 2E_i$ is replaced by E_i.

Finally, we have $L = \mathfrak{so}(2n)$ (type D_n), where the matrices take the same form as in the previous paragraph, except that the first row and column of those matrices are omitted. The dimension of L is now $2n^2 - n$ while the dimension of D is again n; the linear functions on D now arising are now $\pm E_i \pm E_j$ for $1 \leq i, j \leq n, i \neq j$ (so that both E_i and $2E_i$ are omitted).

We have now met all but finitely many of the basic objects of study of the course! We conclude with a construction of the groups corresponding to our Lie algebras, again echoing what is done in manifold theory, but with a new twist. Let A be a finite-dimensional algebra over a field K of characteristic 0 (so that we are free to divide by any nonzero integer in K) and let d be a *nilpotent* derivation on A, so that d^n is the 0 map on A for some n. We can then set up the usual power series definition of the exponential $\exp d = \sum_{i=0}^{n} d^i/i!$, taking d^0 as usual to be the identity map; this makes sense without any completeness assumption on K since it has only finitely many terms. Then the usual formal calculation as in the case $K = \mathbb{R}$ or $\mathbb{C}$, not assuming that d is nilpotent and taking the full power series expansion $\sum_{i=0}^{\infty} d^i/i!$ of $\exp d$, shows that $\exp d$ is an *automorphism* of A: $\exp d(a) \exp d(b) = \exp d(ab)$ for $a, b \in A$. The inverse of $\exp d$ is $\exp -d$. In particular, *$\exp \text{ad } x$ is an automorphism of any Lie algebra L over a field K of characteristic 0 if ad x acts nilpotently on L.* Now it turns out that if K in addition is algebraically closed, then the group generated by all $\exp \text{ad } x$ where ad x is nilpotent can do everything that the group generated by all $\exp \text{ad } x$ for arbitrary x can do if $K = \mathbb{C}$. We call it the *adjoint group* of L and denote it (for mysterious reasons) by Int L. This group will be used at several crucial points in what follows. For now we observe that if $L = \mathfrak{sl}(n, \mathbb{C})$, then its adjoint group G is $PSL(n, \mathbb{C})$ the quotient of group $SL(n, \mathbb{C})$ of $n \times n$ matrices of determinant one by its center, a cyclic group of order n. If $L = \mathfrak{so}(n)$ then $G = PSO(n, \mathbb{C})$, the quotient of the group of $n \times n$ orthogonal matrices of determinant 1 by its center (trivial if n is odd, cyclic of order 2 if n is even). Finally, if $L = \mathfrak{sp}(2n, \mathbb{C})$, then $G = \text{PSp}(2n, \mathbb{C})$, the group of complex symplectic $2n \times 2n$ matrices modulo its center (again of order two).