Lecture 1-7

Welcome to the wonderful world of Lie algebras! I love these objects and hope I can convey to you why they are so exciting. Most of you will have seen Lie algebras before in a course on manifolds; in such a course they are usually introduced as derivations on the algebra of C^∞ real-valued functions on a Lie group, identifying two such functions if they agree on a neighborhood of the identity. Now in this course I will neither assume nor use any manifold theory, but those who have been exposed to such theory will see a couple of echoes of it in this week's lectures (but no further echoes thereafter). Accordingly, let A be an algebra over a field K and let $d : A \to A$ be a *derivation* of A, so that d is linear over K and satisfies the product rule $d(ab) = d(a)b + ad(b)$. The set Der A of all derivations is *not* closed under product (as you might expect), but it *is* closed under the *commutator* or *bracket*: if d and e are derivations, then so is $[de] = de - ed$. Also of course we could define a new operation $[\cdot, \cdot]$ on A itself via $[xy] = xy-yx$; this operation provides a natural measure of the noncommutativity of A. It makes sense, for example, if $A = M(n, \mathbb{C})$, the vector space of $n \times n$ matrices over $\mathbb{C}$; in fact we are primarily interested in derivations on finite-dimensional algebras. Our starting point is the study of this operation (motivated by its appearance as the Lie bracket in manifold theory). To begin with, we observe that the bracket operation is neither commutative nor associative. It still satisfies the distributive law in both arguments, however (that is, it is complex linear in both arguments). In place of the commutative law we have the pair of properties $[x, x] = 0, [x, y] = -[y, x]$; notice that the second of these follows from the first applied to $[x + y, x + y]$. In place of the associative law we have the *Jacobi identity* $[x[yz]] + [y[zx]] + [z[xy]] = 0$, which we may rewrite as $[[xy]z] + [y[xz] = [x[yz]]$; this last equation says exactly that commutation with x, a linear map which we will denote by ad x, satisfies the product rule on brackets. We therefore define a *Lie algebra* L over any field K to be a (finite-dimensional for us) K-vector space endowed with a K-bilinear bracket $[\cdot, \cdot]$ from $L \times L$ to L satisfying $[xx] = 0$ for all $x \in L$ (anticommutativity) and the Jacobi identity. We say that $L' \subset L$ is a (Lie) *subalgebra* of L if it is a K-subspace closed under the bracket; we say that $I \subset L$ is an *ideal* if it is a subalgebra such that $[xy] \in I$ if $x \in I$ (or $y \in I$, by anticommutativity. If I is an ideal of L, then we can make the quotient space L/I into a Lie algebra in the usual way, by decreeing that $[x+I, y+I] = [xy]+I$. The notion of a Lie algebra homomorphism $\pi : L \to L'$ between two Lie algebras L, L' over the same field K is defined in the obvious way; the kernel I of any such map is an ideal of L and if this map is surjective it induces an isomorphism from L/I to L'.

We are naturally most interested in really new examples; that is, in Lie algebras that are *not* associative algebras under any natural product operation, but which admit a bracket operation satisfying the axioms. Probably the easiest of these is $\mathfrak{sl}(n)$, which consists by definition of all $n \times n$ matrices over K of trace 0. Recalling that the trace of the ordinary product AB of any two $n \times n$ matrices equals the trace of BA, we see that $\mathfrak{sl}(n)$ is indeed a Lie subalgebra of the Lie algebra $\mathfrak{gl}(n)$ of all $n \times n$ matrices over K (which we denoted by $M(n, K)$ above). This Lie algebra arises frequently and is said to be of *type* A_{n-1}; the reason for the index shift by -1 will emerge later. We will eventually classify a large and important class of Lie algebras called *semisimple*; this classification will use the letters $A, \ldots, G$ together with numerical subscripts. Thus $\mathfrak{sl}(n)$ is our first example of a semisimple Lie algebra.

Addtional semisimple Lie algebras arise as subalgebras of $\mathfrak{gl}(n)$ (such Lie algebras, by the way, are called *linear*). Let $(\cdot,\cdot)$ be a nondegenerate bilinear form on the vector space K^n that is either symmetric (so that $(v,w)=(w,v)$ for $v,w \in K^n$; the most familiar example is the ordinary dot product), or skew-symmetric (so that $(v,w)=-(w,v)$; here one typically does not see any examples as an undergraduate, but we will construct such forms later this week. They live only in even dimensions $n=2m$.) In all cases we take the set of all $n\times n$ matrices X that are skew-adjoint relative to the form, so that $(Xv,w)=-(v,Xw)$ for all $v,w \subset K^n$. One easily checks that the set of such matrices is not closed under product, but it is closed under commutation, so it is indeed a Lie algebra. If $n=2m+1$ is odd and $(\cdot,\cdot)$ is symmetric, then we get (by definition) the *orthogonal* Lie algebra $\mathfrak{so}(2m+1)$ of *type* B_m; if $(\cdot,\cdot)$ is symmetric but $n=2m$ is even, then we get the orthogonal Lie algebra $\mathfrak{so}(2m)$ of type D_m. These two cases turn out to be different enough that they indeed merit separate labels. The other possibility is that $(\cdot,\cdot)$ is skew-symmetric; in this case $n=2m$ is necessarily even, as mentioned above. The Lie algebra is the *symplectic* one $\mathfrak{sp}(2m)$ of type C_m; oddly enough, in some ways type C_m behaves more like type B_m than type D_m.

More explicitly, we can realize the form $(\cdot,\cdot)$ giving rise to the algebra $\mathfrak{so}(2m+1)$ via the matrix s_m having a 1 as its 11 entry, 0s elsewhere in its first row and column, two blocks, the first a copy of the $m\times m$ zero matrix and the second a copy of the identity matrix I_m immediately below its first row, and then the same two blocks in reverse order just below these blocks; note that s_m is a symmetric $(2m+1)\times(2m+1)$ matrix (see p. 3 of the text). We define the form from the matrix s_m via $(v,w)=v^T\, s_m\, w$, writing w as a column vector and v^T as a row vector. Since s_m is symmetric and invertible the resulting form is indeed symmetric and nondegenerate. The algebra $\mathfrak{so}(2m)$ is defined via the symmetric $2m\times 2m$ matrix s'_m in the same way, where s'_m is obtained from s_m by omitting the first row and column. We will see next time why it is better to use s_m and s'_m than the more obvious choices of I_{2m+1} and I_{2m}, respectively. The algebra $\mathfrak{sp}(2m)$ is defined in the same way via the skew-symmetric matrix s''_m, obtained from s'_m by replacing the copy of I_m in the lower left corner by $-I_m$.