Lecture 1-30

Let H be a maximal toral subalgebra of the semisimple Lie algebra L. Then we have the root space decomposition $L = \oplus_{\alpha \in H^*} L_\alpha$ from last time, where L_α consists of the $x \in L$ such that $[hx] = \alpha(h)x$ for all $h \in H$ and only finitely many root spaces L_α are nonzero. The Jacobi identity shows that $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$; it follows that if $x \in L_\alpha, y \in L_\beta$ and $\alpha + \beta \neq 0$, then ad x ad y acts nilpotently on L so has trace 0. Hence the root spaces L_α, L_β are orthogonal under the Killing form κ whenever $\beta \neq -\alpha$. Since κ is nondegenerate on L, we see that α is a *root* of H in L (that is, $L_\alpha \neq 0$) if and only if $-\alpha$ is a root and the restriction of κ to L_0, the centralizer of H in L, is nondegenerate. We claim that $C = L_0 = H$; clearly $H \subset C$ since H is abelian. If $x \in L$ centralizes H, then so do its semisimple and nilpotent parts x_s, x_n; we must have $x_s \in H$ since H is maximal toral, while $\kappa(x_n, h) = 0$ for all $h \in H$ since ad x_n and ad h are commuting linear maps with ad x_n nilpotent. It follows that κ is nondegenerate on H and the κ-orthogonal complement to H in C is exactly $H_n = \{x_n : x \in H\}$, so that H_n is a subalgebra of L. But then this subalgebra consists of ad-nilpotent elements, whence it acts on L by upper triangular matrices, forcing $\kappa(x, y) = 0$ for all $x, y \in H_n$. Nondegeneracy of κ on C now forces $H_n = 0, C = H$, as desired.

Pulling off the 0-root space H separately, we can rewrite the root space decomposition of L relative to H in its standard form as $L = H \oplus \oplus_{\alpha \in \Phi} L_\alpha$, where Φ consists of all nonzero $\alpha \in H^*$ with $L_\alpha \neq 0$. We call Φ the *root system* of L and its elements *roots*; as defined above Φ depends on the choice of H, but we will later see that it is essentially independent of this choice. Since κ is nondegenerate on H, this form allows us to identify H with H^*; to every $\alpha \in H^*$ there corresponds $t_\alpha \in H$ such that $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in H$. We can also transfer to H^* the restriction of κ to H, decreeing that $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$ for $\alpha, \beta \in H^*$. We have seen that $\alpha \in \Phi$ if and only if $-\alpha \in \Phi$; an easy argument using associativity and nondegeneracy of κ shows that $[L_\alpha, L_{-\alpha}]$ is one-dimensional with basis t_α for all $\alpha \in \Phi$; we cannot have $[L_\alpha L_{-\alpha}] = 0$ since then ad x, ad y would be commuting nilpotent linear maps for $x \in L_\alpha, y \in L_{-\alpha}$, forcing $\kappa(L_\alpha, L_{-\alpha}) = 0$ and contradicting nondegeneracy. Moreover we cannot have $\alpha(t_\alpha) = 0$, for otherwise we could find $x \in L_\alpha, y \in L_{-\alpha}$ with $[xy] = t_\alpha, [t_\alpha x] = [t_\alpha y] = 0$, forcing x, y, t_α to span a solvable subalgebra S of L with $t_\alpha \in [SS], t_\alpha$ nilpotent, contradicting $t_\alpha \in H, t_\alpha \neq 0$. It follows for every nonzero $x_\alpha \in L_\alpha$ there is a nonzero $y_\alpha \in L_{-\alpha}$ with $x_\alpha y_\alpha] = h_\alpha = 2t_\alpha/(\alpha,\alpha)[h_\alpha x_\alpha] = 2x_\alpha, [hy_\alpha] = -2y_\alpha$, so that $x_\alpha, y_\alpha, h_\alpha$ span a subalgebra S_α of L isomorphic to $\mathfrak{sl}(2, K)$. Thus *any semisimple Lie algebra is built up out of copies of* $\mathfrak{sl}(2, K)$; the $\alpha \in \Phi$ span all of H^*, since otherwise some nonzero $h \in H$ would centralize all of L.

Now we are in a position to apply our knowledge of the finite-dimensional representation theory of $\mathfrak{sl}(2, K)$. First let $\alpha \in \Phi$ and consider the sum R_α of H and the root spaces $L_{c\alpha}$ where c runs through the nonzero elements of K with $c\alpha \in \Phi$. Then R_α is a module for the copy S_α of $\mathfrak{sl}(2, K)$ in L constructed in the previous paragraph. The toral subalgebra H accounts for all occurrences of the weight 0 in R_α. The codimension-one subspace Ker α of H is centralized by S_α; the only other occurrence of the weight 0 comes from h_α, which lies in S_α itself. Thus the weight 2 occurs only once in R_α (as that of x_α) and higher even weights do not occur. In particular, we must have $L_{2\alpha} = 0$: *twice a root is never a root.* But then $(1/2)\alpha$ is not a root either; the weights of R_α consists of 0 with multiplicity $\dim H$ and 2 with multiplicity one. It follows that *all root spaces L_α have dimension one and the only multiples of a root α that are roots are $\pm\alpha$.*

Now extend the action of S_α to all of L. If β is a root, then $\beta(h_\alpha)$ is a weight of L relative to S_α, forcing $\beta(h_\alpha) \in \mathbb{Z}$. But now the weights in any finite-dimensional S_α-module occur in pairs $\pm m$, with a weight vector for one of these weights obtained from a weight vector for the other one by bracketing repeatedly with either x_α or y_α. We deduce that *if $\alpha, \beta \in \Phi$, then $\beta - \beta(h_\alpha)\alpha \in \Phi$, with.$\beta(h_\alpha) \in \mathbb{Z}$.*

Finally, let $\alpha_1, \ldots, \alpha_n$ be a basis of H^* consisting of roots. Given $\beta \in \Phi$, write $\beta = \sum_{i=1}^n c_i\alpha_i$; then I claim that all $c_i \in \mathbb{Q}$. To see this, pair both sides of $\beta = \sum c_i\alpha_i$ with any α_j under the form $(\cdot,\cdot)$ and divide by (α_j, α_j). We get a linear system of equations with rational coefficients in the c_i with a unique solution in K; that unique solution must then lie in the copy of $\mathbb{Q}$ inside K. Thus all roots lie in a *rational* subspace of H^* of dimension $\dim_K H^*$. We will wrap up the remaining key properties of roots and look at examples of root systems next time.