Lecture 1-28

As a consequence of Weyl's Theorem, we now show that *semisimple linear Lie algebras L are closed under the Jordan decomposition*; that is, for any matrix x in such a Lie algebra L, the semisimple and nilpotent parts x_s, x_n of x as a matrix also lie in L. (We already know that x has semisimple and nilpotent parts as an element of L, which lie in L, but now we want to see that they coincide with x_s and x_n). To prove this we recall first that ad x_s and ad x_n, being polynomials in ad x without constant term, take L to L, so at least x_s, x_n lie in the normalizer N of L in $\mathfrak{gl}(n, K)$, the ambient matrix algebra containing L. If we knew that $N = L$ we would be done, but unfortunately this is false; scalar matrices lie in N but not in L. We therefore replace N by the subalgebra L' consisting of all matrices y acting with trace 0 on any L-submodule of $V = K^n$. Since $L = [LL]$ consists of trace 0 matrices, we have $L \subset L'$; we now claim that $L = L'$. By Weyl's Theorem we may write L' as the direct sum of L and a complementary submodule M; since $[LL'] \subset L$, by definition of N, we see that L acts trivially on M, whence any matrix y in M acts by a scalar on any irreducible L-submodule of V, by Schur's Lemma. Since y acts with trace 0 on any such submodule and V is the direct sum of such submodules, we conclude that $y = 0$, as desired. In particular, *the semisimple and nilpotent parts x_s, x_n of any element x of a semisimple Lie algebra L continue to act semisimply and nilpotently, respectively, in any finite-dimensional representation of L.*

Continuing with representation theory, let L be the smallest semisimple Lie algebra $\mathfrak{sl}(2, K)$; recall that L has basis h, x, y, where $[hx] = 2x, [hy] = -2y, [xy] = h$. We will now classify the irreducible finite-dimensional L-modules V. We know by above that h acts diagonally on V, so V is the sum of its eigenspaces V_λ (which we call *weight spaces*) under the action of h. These eigenspaces are well known to be linearly independent. Since there are only finitely many of them, there is a weight space V_λ such that $V_{\lambda+2} = 0$. Now the Jacobi identity shows that x, y map V_μ to $V_{\mu+2}, V_{\mu-2}$, respectively (for this reason x and y are sometimes said to act by *raising* and *lowering* operators). Choosing any nonzero $v_0 \in V_\lambda$ and setting $v_i = (1/i!)y^i v_0$ for $i \geq 0$, we can check easily by induction that $hv_i = (\lambda - 2i)v_i, yv_i = (i+1)v_{i+1}, xv_i = (\lambda - i + 1)v_{i-1}$. But now the nonzero v_i are independent, so there must be a least m with $v_{m+1} = 0$. Since xv_{m+1} is a multiple of v_m, this multiple must be 0, forcing $\lambda - m + 1 = 0$: the weight λ, called for obvious reasons the *highest weight* of V, is a nonnegative integer m, one less than the dimension of V (since V is clearly spanned by the nonzero v_i). The full set of weights of V is then $m, m-2, m-4, \ldots, -m$, so clearly V has the weight 0 or 1, with multiplicity one, but not both. The "natural representation" $V = K^2$ has weights $1, -1$; the adjoint representation L has weights $2, 0, -2$. A general finite-dimensional representation of L is a direct sum of irreducible ones; you will show in HW that an irreducible representation exists of every possible dimension m. So the representation theory of L is only slightly more complicated than that of a finite group over $\mathbb{C}$; there are infinitely many irreducible representations up to isomorphism, but they are very well controlled.

Staying with $L = \mathfrak{sl}(2, K)$ just a little longer, we now look at its adjoint group. In terms of matrix units, we have $h = e_{11} - e_{22}, x = e_{12}, y = e_{21}$. Computing the product of the exponentials of ax, by, and ax, in that order, for any $a, b \in K$ with $ab = -1$, we get a matrix y_c with first row $(0, c)$ and second row $(-c^{-1}, 0)$ for some $c \in K$; multiplying this by the corresponding matrix with c replaced by $-c^{-1}$, we get a diagonal matrix with diagonal entries c^2, c^{-2}. Since K is algebraically closed, we deduce from the theory of row operations that *any matrix of determinant 1 is the product of exponentials of nilpotent elements in* L, so that indeed Int $L = PSL(2, K)$, as mentioned previously. The group $SL(2, K)$ acts in a natural way on any finite-dimensional L-module M; this action descends to an action of the adjoint group $PSL(2, K)$ if and only if the action of the scalar matrix $-I$ is trivial. If M is irreducible, this happens if and only if M has odd dimension. In general the action of the matrix y_c defined above with $c = 1$ interchanges positive and negative weight spaces in M, just as it interchanges the elements x and y in L.

Now let L be an arbitrary semisimple Lie algebra (over our usual algebraically closed field of characteristic 0). If L consisted of (ad-)nilpotent elements, it would be nilpotent, by Engel's Theorem; since this is not the case, but L is closed under Jordan decomposition, there must be a nonzero subalgebra of L consisting of (ad-)semisimple elements. Call such a subalgebra *toral*. Now a toral subalgebra must be abelian, for otherwise it would contain elements x, y such that $[xy] = ky$ for some $k \in K, k \neq 0$; but then the bracket of y and x would be a sum of eigenvectors for ad y corresponding to nonzero eigenvalues, if any, a contradiction. Now fix a maximal toral subalgebra H (not contained in any other). Then H acts on L by a commuting family of semisimple maps; by a standard result of linear algebra, there must a fixed basis of L with respect to which all $h \in H$ act diagonally via the bracket. Equivalently, there is a finite set S of linear functions on H such that L is the direct sum of its simultaneous eigenspaces L_α as α runs over S. This is called the *root space decomposition* of L and will serve as our key tool in classifying all such Lie algebras L up to isomorphism.