Lecture 1-25

So far we have stuck to the structure theory of Lie algebras, giving no more than the first definitions of representation (or module) theory. Now however we must digress to prove a basic result in the latter theory, which we will need to prove the deeper results in structure theory. This is *Weyl's Theorem*, which asserts that any finite-dimensional module M over a semisimple Lie algebra L is completely reducible, that is, it is a direct sum of irreducible modules; recall that the same result holds for finite-dimensional representations of a finite group over a field of characteristic 0.

The proof will require some preparation; we begin by describing a couple of ways to make new modules over a Lie algebra L from old ones. If V, W are two L-modules, then we can take the tensor product $V \otimes_K W$ of V and W as K-vector spaces; recall that the dimension of the tensor product is the product of the dimensions of V and W. If V and W were modules over a group G, then we could define a natural G-action on $V \otimes W$ via the recipe $g(v \otimes w) = gv \otimes gw$ for $g \in G, v \in V, w \in W$. For Lie algebras we must "differentiate" this recipe: we decree that $x(v \otimes w) = xv \otimes w + v \otimes xw$ for $x \in L, v \in V, w \in W$. A simple calculation then shows that the difference of the actions by xy and yx equals the action of the bracket $[xy]$, as required. In a similar way, the set $\hom_K(V, W)$ of K-linear maps from V to W has a module structure, defined by $(xf)(v) = xf(v) - f(xv)$ for $x \in L, v \in V, f \in \hom(V, W)$. In particular, L acts trivially (i.e. by 0) on $f \in \hom(V, W)$ if and only if f is an L-module homomorphism (commutes with the action of L). We also need *Schur's Lemma*, which says that the only module homomorphisms from an irreducible finite-dimensional L-module M to itself are the scalars; this follows since any eigenspace of such a homomorphism must be submodule, hence either all of M or 0, so that for a suitable eigenvalue it must be all of M.

Finally we need *Casimir elements*. These are most naturally attached to representations rather than to modules, that is, to Lie algebra homomorphisms $\phi : L \to \mathfrak{gl}(m, K)$ for some m; let $V = K^m$ be the corresponding L-module. Assume that L is semisimple. The form $(\cdot, \cdot)$ on L defined via $(x, y) = \mathrm{tr}(\phi(x)\phi(y))$ is associative, for the same reason that the Killing form is associative, and its radical is a solvable ideal of L by Cartan's Criterion, so this radical is 0 and the form is nondegenerate. Fix a basis $x_1, \ldots, x_n$ of L and let $y_1, \ldots, y_n$ be the dual basis relative to the form, so that $(x_i, y_j) = \delta_{ij}$, the Kronecker delta. The Casimir element c_ϕ corresponding to ϕ is then the matrix $\sum_{i=1}^n \phi(x_i)\phi(y_i)$. For $x \in L$ write $[xx_i = \sum_j a_{ij} x_j, [xy_i] = \sum_j b_{ij} y_j$. Associativity of the form implies that $([x_i x], y_j) = (x_i, [xy_j]$ for all i, j, whence the coefficients satisfy $a_{ij} = -b_{ji}$ for all i, j. For any $x \in L$ the commutator $[\phi(x) c_\phi]$ of $\phi(x)$ and c_ϕ, which equals $\sum_{i=1}^n ([\phi(x)\phi(x_i)]\phi(y_i) + \phi(x_i)[\phi(x)\phi(y_i)])$. The coefficient of $\phi(x_i)\phi(y_j)$ in this sum is $a_{ij} + b_{ji} = 0$ for all i, j, so c_ϕ *commutes with all* $\phi(x)$. Hence c_ϕ must be a scalar matrix if V is irreducible. Since its trace is easily seen to be $\dim L$ (by the way the dual bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ were chosen), it must in fact be the scalar $n/m = (\dim L)/(\dim V)$; in particular, it does not depend on the choice of basis $x_1, \ldots, x_n$ of L. (Actually, it is not difficult to see in general that c_ϕ is independent of the choice of basis.)

Now we are ready to tackle the proof of Weyl's Theorem. I warn you that the argument in the text is garbled at a couple of crucial places. Given a submodule N of an L-module M we must show that N has a complement, that is, there is another submodule N' such that M is the direct sum of N and N'. (This is what Humphreys should have said at the very beginning of his proof.) We first prove this if N has codimension one, by induction on $\dim M$. If N is reducible, say with proper submodule P, then the quotient module N/P is naturally a submodule of M/P of codimension one, whence by hypothesis it has a complementary submodule $\bar{Q}$. Pulling back to M we get a submodule Q of M containing P as a submodule of codimension one. By inductive hypothesis again, P has a one-dimensional complement in Q, which is also a complement to N in M, as desired. Thus we are reduced to the case where N is irreducible. The Casimir element c of the representation corresponding to M then acts by the nonzero scalar $\dim L/\dim N$, by a calculation in the last paragraph, but both L and c must act by 0 on the quotient module M/N, since this quotient has dimension one and $L = [LL]$. Hence the the kernel N' of c on M furnishes the one-dimensional complement N' to N we are looking for.

Thus submodules of finite-dimensional modules over a semisimple Lie algebra of codimension one always have complements. Now suppose that N is any proper submodule of M. Then L acts on the vector space $\hom(M, N)$, preserving the subspace H consisting of all linear maps whose restriction to N is a scalar. This submodule H then admits the submodule H' consisting of all linear maps whose restriction to N is 0. If x spans a one-dimensional complement to H' in H, then L acts on H' trivially as above, since $L = [LL]$, so that x is an L-module homomorphism from M to N restricting to a nonzero scalar on N. Its kernel N' is the complement to N we are looking for.

Weyl's Theorem was originally proved analytically, using integration on compact Lie groups; Brauer later gave the proof that we have followed. The same result (that finite-dimensional modules are completely reducible) of course also holds for finite groups over fields of characteristic 0 and semisimple Artinian rings; for simple Artinian rings R we have the further bonus that there is only one irreducible R-module up to isomorphism: R is isomorphic to the ring $M_n(D)$ of $n \times n$ matrices over a division ring D, and then its unique irreducible (left) module is D^n, the space of column vectors with coordinates in D.