Lecture 1-23

Continuing from last time, we now give a *Cartan's Criterion* for a Lie algebra over an algebraically closed field of characteristic 0 to be solvable, again purely in terms of traces. We state this first for linear Lie algebras and then generalize (as usual via the adjoint representation) to arbitrary Lie algebras. In the linear setting, *if L is a Lie subalgebra of $\mathfrak{gl}(n,K)$ such that tr $xy = 0$ for all $x \in [LL], y \in L$, then L is solvable.* To prove this it suffices to show that $[LL]$ is nilpotent, which will follow if we can show that the matrices in $[LL]$ are nilpotent. Setting $M = \{x \in \mathfrak{gl}(n,K) : [xL] \subset [LL]\}$, we need to show (thanks to the criterion last time) that tr $xy = 0$ for all $x \in [LL], y \in M$; by hypothesis this holds if $x \in [LL], y \in L$. But now if $x, y \in L, z \in M$, then tr $[xy]z =$tr $x[yz] = 0$ by associativity, since $[yz] \in [LL]$, as desired. More generally, if L is any Lie algebra with $\kappa(x,y) =$tr (ad x ad y) $= 0$ for all $x \in [LL], y \in L$, then the previous result shows that L/Z is solvable, Z the center of L, whence as usual L is also solvable.

For the rest of the course we will be concentrating on Lie algebras that are the polar opposite of solvable, namely semisimple Lie algebras. Recall that the official definition is that L is semisimple if it has no nonzero solvable ideals. In view of the trace criteria we have been working with, it is not surprising that *L is semisimple if and only if its Killing form κ is nondegenerate* (so that the only $x \in L$ with $\kappa(x,y) = 0$ for all $y \in L$ is $x = 0$). To prove this, note first that if I is an ideal of L, then the Killing form on I, regarded as a Lie algebra, is just the restriction of the Killing form on L to $I \times I$; this follows by extending a basis of I to one of L, since then (ad x)(ad y) maps L to I and the basis vectors in L but not in I make no contribution to the trace of this map. Now associativity of the κ shows that its radical S (consisting of all $x \in L$ with $\kappa(x,y) = 0$ for all $y \in L$) is an ideal of L; by our first remark and Cartan's Criterion, S is solvable, so if L is semisimple we must have $S = 0$. Conversely, if $S = 0$ but L has a nonzero solvable ideal I, then the last nonzero term of the derived series for I is a nonzero abelian ideal J of L, and if $x \in L, y \in J$, then (ad x)(ad y)^2 maps L into $[JJ] = 0$, whence $J \subset S$, a contradiction.

Nondegeneracy of the Killing form for a semisimple Lie algebra is the key result making the classification of such algebras tractable (for example, there is no analogue of the Killing form for finite simple groups, whose classification takes some ten thousand pages). If L is semisimple and I is an ideal of L then its orthogonal complement $I^\perp$, consisting of all $x \in L$ orthogonal to I under κ, is again an ideal by associativity; moreover, $I \cap I^\perp = 0$ by Cartan's Criterion. Hence L is the direct sum of I and $I^\perp$. Continuing in this way, we see that *any semisimple Lie algebra is a finite direct sum of simple Lie algebras*; conversely, it is easy to see that any finite direct sum of simple Lie algebras has no nonzero solvable ideals, so is semisimple (recall that by definition any simple Lie algebra has dimension at least two). Thus the two definitions of semisimplicity given earlier are indeed equivalent.

As a further consequence of nondegeneracy, let L be a Lie algebra and let δ be a derivation of L. The definition of derivation shows that the commutator $[\delta,$ad $x] =$ad δx is a derivation of the form ad y; that is, it is by definition *inner*. Now *if L is semisimple then any derivation of it is inner*. To prove this we note first that $M =$ad L by above is an ideal of the Lie algebra D of derivations of L; this ideal is isomorphic to L since the center of L is trivial. The orthogonal $I = M^\perp$ to M in D relative to the Killing form on D then intersects M trivially, since the Killing form on M is nondegenerate, so the commutator

$[IM]$ of I and M in D is 0. This says exactly that ad $\delta x = 0$ for all $\delta \in I, x \in L$, forcing $I = 0, D = M$, as desired.

We showed last time that the semisimple and nilpotent parts d_s, d_n of any derivation d on any Lie algebra L are again derivations. If L is semisimple, these derivations must be inner, so for any $x \in L$ we get unique $x_s, x_n \in L$ such that $x = x_s + x_n, [x_s x_n] = 0$, and ad x_s, ad x_n act semisimply and nilpotently, respectively, on L. Thus Jordan decompositions exist for any semisimple Lie algebra, even if it is not given as a linear Lie algebra. But what if it is? More precisely, if $L \subset \mathfrak{gl}(n, K)$ is a linear semisimple Lie algebra, then the matrices x in it have semisimple and nilpotent parts x_s, x_n in $\mathfrak{gl}(n, K)$; how do we know that these lie in L? We will see later that indeed they do; for now we just note that they do if $L = \mathfrak{sl}(n, K)$, for then commutation by a semisimple matrix, being the sum of two commuting semisimple matrices, is again semisimple, and similarly for a nilpotent matrix, so ad $x =$ad $x_s +$ ad x_n is the Jordan decomposition of ad x in L if $x_s + x_n$ is the Jordan decomposition of x as a matrix.