Lecture 1-18

We have previously mentioned the Jordan canonical form of a matrix over an algebraically closed field K (of any characteristic), in the context of nilpotent Lie subalgebras of $\mathfrak{gl}(n, K)$. It turns out that this form and an additive decomposition corresponding to it play crucial roles in the theory of Lie algebras, so we digress a bit to discuss this form in more detail.

The Jordan canonical form y of an $n \times n$ matrix x over K is a block diagonal matrix for which the blocks are upper triangular with equal entries on the main diagonal, ones on the diagonal above it, and zeroes elsewhere. Given a matrix y in such a form, let y_s be the matrix whose main diagonal is the same as that of y while its other entries are 0 and let $y_n = y - y_s$. Then y_s is semisimple(=diagonalizable), y_n is nilpotent (being strictly upper triangular), and one easily checks that y_y, y_n both commute with y. More generally, let x as above be any $n \times n$ matrix over K and let T be the corresponding linear transformation of K^n, sending v to xv. Let $p(\lambda) = \prod(\lambda - a_i)^{n_i}$ be the characteristic polynomial of x and T. Then it is well known that K^n is the direct sum of the generalized eigenspaces $V_i = \text{Ker } (T - a_i)^{n_i}$. By the Chinese Remainder Theorem there is a polynomial $q(\lambda)$ such that $q(\lambda) \equiv a_i \mod (\lambda - a_i)^{n_i}$ for every i while $q(T) \equiv 0 \mod \lambda$; note that the last congruence is superfluous if one of the a_i is 0, while otherwise the moduli are indeed relatively prime. Then $q(T)$ acts by a_i on each subspace V_i while the difference $p(T) = T - q(T)$ acts nilpotently on each V_i, hence on all of K^n. This says that the matrices x_s, x_n of $q(T), p(T)$ are semisimple and nilpotent, respectively, each commutes with x, and x is their sum; moreover $q(T), p(T)$ are polynomials in T constant term 0. Conversely, if $x = x'_s + x'_n$ is another decomposition of x into semisimple and nilpotent matrices commuting with x, then x'_s and x'_n also commute with x_s and x_n (the latter being polynomials in x), whence $x_s - x'_s$ is again semisimple and $x_n - x'_n$ is again nilpotent. The equality $x_s - x'_s = x'_n - x_n$ then forces $x_x = x'_s, x_n = x'_n$. Thus *any $n \times n$ matrix x over K is uniquely the sum $x_s + x_n$ of semisimple and nilpotent matrices commuting with itself.* This is the *Jordan decomposition* mentioned above, which goes hand in hand with the Jordan form. We call x_s and x_n the semisimple and nilpotent parts of x, respectively.

What makes the Jordan decomposition useful is that it is preserved when one linear Lie algebra is sent to another by a homomorphism. First of all, if $x \in L = \mathfrak{gl}(n, K)$, then ad x_s and ad x_n are respectively the semisimple and nilpotent parts of ad x in the space of linear transformations from $\mathfrak{gl}(n, K)$ to itself; this follows since we have already seen that ad x_s is semisimple, while ad x_n is nilpotent since it is the difference between left and right multiplication by x_n and both such multiplications are nilpotent. Since the commutator of ad x_s and ad x_n is ad $[x_s, x_n]$, this commutator is 0, and ad x_s+ ad x_n is indeed the Jordan decomposition of ad x. Secondly, given a finite-dimensional not necessarily associative algebra A over K, the semisimple and nilpotent parts d_s, d_n of any derivation of A, considered as a linear transformation from A to itself, are again derivations; this follows since the generalized eigenspaces A_a, A_b of A corresponding to the eigenvalues a, b of d are such that $A_a A_b \subset A_{a+b}$, as shown in the text on p. 19, whence the transformation d_s acting by the scalar a on each A_a is a derivation, as desired.

Now we are ready to give a criterion for a transformation x to be nilpotent, stated purely in terms of traces. Assume now that K has characteristic 0 (in addition to be being as usual algebraically closed). *Let $A \subset B$ be two subspaces of $\mathfrak{gl}(n,K)$ and set $M = \{x \in \mathfrak{gl}(n,K) : [x,B] \subset A\}$. Suppose that $x \in M$ is such that the trace tr $xy = 0$ for all $y \in M$. Then x is nilpotent.* To prove this, let $s + n = x_s + x_n$ be the Jordan decomposition of x. Let $a_1, \ldots, a_m$ be the eigenvalues of x and s, counted with multiplicities. We must show that all a_i are 0, or equivalently that the $\mathbb{Q}$-subspace E of K spanned by the a_i is 0. For this it is enough if we can show that any $\mathbb{Q}$-linear map f from this subspace to $\mathbb{Q}$ is 0. Given f, let $r(\lambda)$ be a polynomial in $K[\lambda]$ with constant term 0 such that $r(a_i - a_j) = f(a_i) - f(a_j)$ for all i, j; there is no ambiguity in the assigned values since if $a_i - a_j = a_k - a_\ell$, then by linearity $f(a_i) - f(a_j) = f(a_k) - f(a_\ell)$. Now we know that ad s acts diagonally on $\mathfrak{gl}(n,K)$ with eigenvalues $a_i - a_j$, so the diagonal matrix y with eigenvalues $f(a_1), \ldots, f(a_m)$ (and the same eigenvectors as s) is such that ad $y = r(\text{ad } s)$. But now ad s is the semisimple part of ad x and can be written as polynomial in ad x with constant term 0, so ad y can also be written as such a polynomial, whence ad y maps B into A and $y \in M$ and tr $xy = 0$. But this last trace is $\sum_i a_i f(a_i)$; applying f, we get $\sum f(a_i)^2 = 0$, whence all $f(a_i) = 0$, as desired.

We want to apply this criterion to give a criterion in terms of traces for a Lie algebra to be solvable. To do this note first that for any $x, y, z \in \mathfrak{gl}(n,K)$ we have tr $([xy]z =$ tr $(x[yz]$, since $[xy]z = xyz - yxz, x[yz] = xyz - xzy$ and tr $xzy =$ tr zyx. Define the *Killing form* κ on a Lie algebra L via $\kappa(x,y) =$ tr (ad x,ad y). Then κ is said to be *associative* or *invariant*, in the sense that $\kappa([xy],z) = \kappa(x,[yz])$. We will explore the powerful consequences of this definition next time.