Lecture 1-16

Following the lines of our earlier investigation of linear nilpotent Lie algebras, we now look at solvable Lie subalgebras of $\mathfrak{gl}(m, K)$. Here for the first time we must make two previously mentioned assumptions on K, namely that it is algebraically closed and of characteristic 0. The result, called Lie's Theorem, states that *given any solvable Lie subalgebra L of $\mathfrak{gl}(m, K)$ with K algebraically closed and of characteristic 0, there is a common eigenvector $v \in K^m$ for the matrices in L.* (Note that, in contrast to the lemma used to prove Engel's Theorem, we make no assumption in advance about the matrices in L, but we do assume something about the structure of L itself.)

To prove this we follow the approach in proving the lemma just mentioned and argue by induction on the dimension d of L. If $d = 1$, so that L is spanned by a single matrix x, then any eigenvector of x has the desired property (but notice that we have already had to use our assumption that K is algebraically closed to be sure that x has an eigenvector). If the result holds for solvable Lie algebras of dimension less than d, then we note by the definition of solvability that L must properly contain its solvable derived subalgebra $[LL]$, and any subspace I of L containing $[LL]$ is automatically an ideal, so L has a solvable ideal I of codimension one, so that the matrices in I have a common eigenvector. This implies that for some linear function λ from I to K the common λ-eigenspace $V = V_\lambda = \{v \in K^n : xv = \lambda(x)v, x \in I\}$ is nonzero. Now if we can show that the matrices in L leave V invariant, then we will be done, for L is spanned by I and just one matrix y, so that any eigenvector of y in V does the job. To show that L preserves V, we let $x \in L, v \in V, y \in I$ and look at $xyv = yxv + [xy]v$. We have $xyv = \lambda(y)xv$ and $[xy] \in I$, so we must show that $\lambda[xy] = 0$. Let $n > 0$ be the smallest integer such that $v, xv, \ldots, x^n v$ are linearly dependent and let V_i be the span of $v \ldots, x^{i-1}v$ for $i > 0$, while $V_0 = 0$, so that x maps V_n to itself. Then we have $yx^i v = yxx^{-1}v = xyx^{-1}v - [yx]x^{i-1}v$, whence by induction $yx^i v \equiv \lambda(y)x^i v \mod V_i$ for all i. Hence y acts linearly on W_n by an upper triangular matrix with $\lambda(y)$ on the diagonal and its trace on W_n is $n\lambda(y)$. Then $[xy]$ acts as the commutator of two matrices on W_n, so its trace there must be 0. Since the characteristic of K is 0, we get $\lambda[xy] = 0$, as desired.

Given this result, we can form the quotient space K^m/Kv, on which L acts linearly, so L has a common eigenvector in this quotient space. Iterating this construction, we see that under the hypothesis the vector space K^m admits a chain of subspaces $K_0 = 0 \subset K_1 \subset \ldots$ preserved by L such that $\dim K_i = i, K_m = K^m$. Equivalently, *there is a bsis of K^m such that the matrices in L are all upper triangular with respect to this basis.* In particular, applying this result to the adjoint action of L on itself, we see that *any solvable Lie algebra over an algebraically closed field K of characteristic 0 admits a chain of ideals $L) = 0 \subset L_1 \subset \ldots$ such that $\dim L_i = i, L_n = L$ for some n.* Also *$[LL]$ is nilpotent whenever l is solvable*, since $[LL]$ modulo its center acts on itself by strictly upper triangular and thus nilpotent matrices. It turns out that both of these properties can fail for a solvable Lie algebra L over a field K which is not algebraically closed of characteristic 0. Nevertheless, solvable Lie algebras L admitting a chain of ideals $L_0 \subset L_1 \subset \ldots$ of this type are important enough to deserve a special name; they are called *completely solvable.*

If L is a nilpotent subalgebra of $\mathfrak{gl}(m, K)$ and K as usual is algebraically closed of characteristic 0, then we can say more: there is a basis of K^m such that the matrices in L

are block diagonal with each block upper triangular with equal entries along its diagonal. This is a Lie algebra analogue of the Jordan canonical form for a single matrix over K; it turns out that this form will be a basic tool in our study of Lie algebras as well.

Recall from Friday's lecture that a *module* M over a $K-$Lie algebra is a K-vector space such that for all $x \in L, m \in M$ we have $xm \in M$ depending linearly on x and m such that $xym - yxm = [xy]m$ for all $x, y \in L, m \in M$. Then n-dimensional modules M over a Lie algebra L over K correspond to Lie algebra homomorphisms from L into $\mathfrak{gl}(n, K)$; in this case, as previously noted we call either M or the homomorphism a representation of L. The notion of representation makes perfect sense if M is infinite-dimensional as well, and in fact infinite-dimensional representations of a finite-dimensional Lie algebra are actively studied in current research.

Just as in the ring case, we call a subspace N of an L-module M an *L-submodule* if it is preserved by L; if so then L acts in a natural way on the quotient space M/N, which accordingly is called a quotient L-module. We say that M is *simple* or *irreducible* if it has no submodules apart from 0 and M itself. In this language Lie's Theorem says that the only irreducible finite-dimensional modules over a solvable Lie algebra L have dimension one (assuming as usual that the basefield K is algebraically closed and of characteristic 0).