Lecture 1-14

We have seen that the Lie algebra $\mathfrak{u}(n)$ of $n \times n$ strictly upper triangular matrices over a field K, which consists of nilpotent matrices (their nth powers are 0), is nilpotent as a Lie algebra. It is a remarkable fact that *any Lie algebra L consisting of nilpotent $n \times n$ matrices is conjugate to a subalgebra of $\mathfrak{u}(n)$ and so in particular is nilpotent.* To prove this it is enough to show that for any such L there is a nonzero $v \in K^n$ with $xv = 0$ for all $x \in L$. For if so then L also acts by nilpotent matrices on the quotient vector space K^n/Kv, whence there is a nonzero $\bar{v}_2 \in K^n/Kv$ with $x\bar{v}_2 = 0$ for all $x \in L$; pulling $\bar{v}_2$ back to $v_2 \in K^n$ we get a vector v_2 such that any $x \in L$ sends v to 0 and v_2 to a multiple of v. Continuing in this way, we get a basis $v, v_2, \ldots, v_n$ of K^n such that the matrices in L with respect to this basis are strictly upper triangular, as desired.

We show that v exists by induction on the dimension d of L. If $d = 1$, so that L is spanned by a single matrix x, then there is k with $x^k = 0 \neq x^{k-1}$, whence any nonzero vector in $x^{k-1}K^n$ does the trick. In general let L' be any proper subalgebra of L of maximal dimension; such a subalgebra must exist since any one-dimensional subspace of L is a subalgebra. Now for any $x \in L$, the transformation ad x from L to itself is nilpotent, for it is the difference of left and right multiplication by x and these two nilpotent transformations commute. Hence L acts on the quotient space L/L' by nilpotent matrices, so that by inductive hypothesis there is $x \in L, x \notin L'$ with $[Lx] \subset L'$ and $L' + Kx$ is a subalgebra of L. By maximality we must have $L' + Kx = L$ and L' is an ideal of L. By the inductive hypothesis the subspace V of K^n consisting of all w with $yw = 0$ for all $y \in L'$ is nonzero; but now a simple calculation using the Jacobi identity shows that x sends W to itself, so acts nilpotently on W. By the argument in the base case, there is $v \in W$ with $xv = 0$, whence we have $zv = 0$ for all $z \in L = L' + Kx$ as desired. In particular, if L is a nilpotent Lie algebra, then it acts on itself by nilpotent matrices, via the adjoint representation, so we deduce from the above result that there is a chain $L_0 = 0 \subset L_1 \subset \cdots \subset L_n = L$ of ideals of L such that that $\dim L_i = i$ and $[LL_i] \subset L_{i-1}$ for all i; equivalently, L acts trivially on the one-dimensional quotient L_i/L_{i-1}.

A famous example of a nilpotent Lie algebra is the *Heisenberg algebra* of strictly upper triangular 3×3 matrices over a field K. This algebra is three-dimensional, being spanned by the matrix units $x = e_{12}, y = e_{23}$, and $z = e_{13}$; we have $[xy] = z, [xz] = [yz] = 0$. Last time we observed that the the algebra $L = \mathfrak{sl}(2)$ is simple over any field K of characteristic not equal to 2; by contrast, if the characteristic of K is 2, then L is nilpotent and in fact isomorphic to the Heisenberg algebra: setting $x = e_{12}, y = e_{21}, z = e_{11} - e_{22} = I$, we find that x, y, z satisfy the bracket relations given above. This is our first indication that the characteristic of the basefield K can be important; for virtually all of the theory that follows, we will need to assume that K is in fact algebraically closed and of characteristic 0.

Now let L be any Lie subalgebra of $\mathfrak{gl}(n)$ and suppose that ad x, as a linear transformation from L to itself, is nilpotent for all $x \in L$. Then *Engel's Theorem* asserts that L is nilpotent. To see this we note that the adjoint representation realizes the quotient L/Z of L by its center Z as a Lie algebra of nilpotent matrices, whence L/Z is nilpotent; but then so is L, by a remark made last time.

Thus nilpotence is a "local" property of Lie algebras in the sense that if it holds for every element of a Lie algebra then it holds for the Lie algebra itself. A famous problem in group theory called *Burnside's problem* asks whether the same is true of the property of having finite order. More precisely, let G be a group generated by two elements such that there is a positive integer n with $g^n = 1$ for all $g \in G$. Must G be finite? The answer is yes for $n = 2, 3, 4$, or 6, but no in general (to show how hard this problem is, it is still not known what the answer is for $n = 5$). Less ambitiously, one might ask whether among finite groups generated by two elements with every element having order dividing n there is a largest group. Here the answer turns out to be yes; this was proved by Efim Zelmanov, who later received the Fields Medal for his work. (It was announced while Zelmanov and I were both at Yale, in 1990, just before I came here). The connection with Lie algebras is that *if a finitely generated Lie algebra L has $(\text{ad } x)^n = 0$ for fixed n and all $x \in L$, then L is nilpotent*; Zelmanov proved this result at the same time as the group-theoretic one.