Lecture 1-11

We begin with a few more remarks about the group Int L defined last time. Assume for a moment that the basefield K is $\mathbb{R}$ or $\mathbb{C}$ and let M be an $n \times n$ matrix over K. Then it is well known that the series $e^M = \sum_{i=0}^{\infty} M^i/i!$ always converges (that is, the separate series for each entry in the matrix power series all converge) and that $e^M e^{-M} = I$. Now consider $A =$ad M, the linear function from the space of $n \times n$ matrices over K to itself send a matrix N to $MN - NM$. Then A is the difference between two commuting linear maps, namely left and right multiplication by M. Forming the series $\sum_{i=0}^{\infty} A^i/i!$, we find by a formal calculation that it sends any matrix N to $e^M N e^{-M}$. Thus the adjoint group Int L of any classical Lie algebra L acts on L by conjugation of matrices; the matrices involved have determinant one in all cases and in addition preserve the bilinear form $(\cdot, \cdot)$ in types B, C, and D. Since conjugation by any scalar matrix is trivial, this is why we must mod out by the scalar matrices in the formulas for Int L in the classical cases.

We now develop the theory of general Lie algebras. An easy example of an ideal in any Lie algebra L over a field K is the *derived algebra* $[LL]$, which is by definition spanned by all $[x, y]$ as x, y range over L (it is not true in general that the set of all brackets $[xy]$ is closed under addition, just as the set of commutators $ghg^{-1}h^{-1}$ as g, h range over a group G need not be a subgroup of G; in both cases we close under the relevant operation to get the subalgebra or subgroup). More generally, if I, J are ideals of L, then the Jacobi identity shows that $[IJ]$, spanned by all brackets $[xy]$ as x runs over I and y over J, is an ideal. Another ideal of L is its *center* $Z(L)$, consisting of all $x \in L$ such that $[xy] = 0$ for all $y \in L$. We say that L is *simple* if its dimension is larger than 1 and L has no ideals apart form itself and 0 (any 1-dimensional Lie algebra is necessarily abelian and satisfies this condition, but we do not want to call it simple). We say (for now) that L is *semisimple* if it is a finite direct sum $\oplus_i L_i$ of simple ideals L_i, so that L is the direct sum of the L_i as a vector space, each L_i is simple as a Lie algebra, and $[L_i L_j] = 0$ if $i \neq j$; this is not the official definition but will later be shown to be equivalent to it. In particular, if L is simple or semisimple, we must have $L = [LL], Z(L) = 0$. Whenever $Z(L) = 0, L$ is isomorphic to a linear Lie algebra, that is, to a Lie algebra of matrices. To see this, note that the map from L to $\mathfrak{gl}(L) \cong M_n(K)$ (i.e. to the set of linear maps from the K-vector space L to itself) sending x to ad x is a Lie algebra homomorphism, by the Jacobi identity; its kernel is clearly $Z(L)$, so L is isomorphic to a subalgebra of $M_n(K)$ if $Z(L) = 0$. More generally, any Lie algebra homomorphism from L to some $M_m(K)$ is called a *representation* of L; this definition parallels that of a representation of a group. The corresponding notion of *L-module* is a finite-dimensional K-vector space M such that for every $x \in L, m \in M$ there is $x.m \in M$ such that $(x + y).m = x.m + y.m, x.(m_1 + m_2) = x.m_1 + x.m_2, kx.m = x.km = k(x.m$ if $k \in K$, and finally $x.y.m - y.x.m = [xy].m$ for $x, y \in L, m \in M$.

As a challenging exercise, try to prove directly that $L = \mathfrak{sl}(n)$ is a simple Lie algebra for $n > 1$, over any field K of characteristic 0. The case $n = 2$, where L has basis h, x, y, where $h = e_{11} - e_{22}, x = e_{12}, y = e_{21}$, is worked out in the text in §2.1; here $[hx] = 2x, [hy] = -2y$, and $[xy] = h$. In general one shows that every nonzero ideal of $L = \mathfrak{sl}(n)$ contains first one matrix unit e_{ij} for $i \neq j$, then all such units, and finally all of L. The algebra $\mathfrak{gl}(n)$ of all $n \times n$ matrices over K is almost simple, but not quite; it has the ideals L and the subspace $KI = Z(\mathfrak{gl}(n))$ of all scalar matrices.

While L has no proper ideals, certain subalgebras of it have many such ideals. Specifically, the subalgebra $\mathfrak{t}(n)$ of upper triangular matrices has the ideal $\mathfrak{u}(n) = [\mathfrak{t}(n), \mathfrak{t}(n)]$ of strictly upper triangular matrices (with zeroes on the diagonal) as an ideal; in fact, note that $\mathfrak{t}(n)$ is also an associative K-algebra under multiplication and $\mathfrak{u}(n)$ is a two-sided associative ideal of it. The quotient $\mathfrak{t}(n)/\mathfrak{u}(n)$, either of Lie algebras or of associative algebras, is naturally isomorphic to the (Lie or associative) algebra $D = \mathfrak{d}(n)$ of diagonal matrices, and D is abelian whether it is regarded as a Lie or associative algebra. Passing to the derived subalgebra $[\mathfrak{u}(n), \mathfrak{u}(n)]$ of $\mathfrak{u}(n)$, we find that it consists of all strictly upper triangular matrices whose superdiagonals (immediately above the main diagonal) are also 0; the quotient of $\mathfrak{u}(n)$ by its derived algebra is again abelian. Iterating the derived subalgebra construction, we find that we eventually get the 0 algebra. More generally, given an arbitrary Lie algebra L, we define its *derived series* $L^{(0)}, L^{(1)}, \ldots$ via $L^{(0)} = L, L^{(n)} = [LL^{(n-1)}$; if we then have $L^{(n)} = 0$ for some n, then we call L *solvable*. This notion is motivated by and is strongly analogous to a parallel notion for groups which was defined much earlier, in fact by Galois. There are also parallel notions of something called *nilpotence* for both Lie algebras and groups, but in that case the notion for Lie algebras was defined first. We define the *lower central series* of an arbitrary Lie algebra L by $L^0 = L, L^n = [LL^{n-1}]$; we say that L is nilpotent if $L^n = 0$ for some n. Equivalently, L is nilpotent if and only if for some n all n-fold brackets $[\ldots[x_1 x_2]x_3]\ldots x_n]$ are 0 in L Clearly any nilpotent Lie algebra is solvable, but the converse fails, since $\mathfrak{t}(n)$ is solvable but not nilpotent. (The notions of solvability and nilpotence for groups G are defined in the same way, defining $[GG]$ as the commutator subgroup of G.)

It is easy to check that ideals and homomorphic images of solvable Lie algebras are again solvable; moreover, if I is a solvable ideal such that the quotient L/I is solvable, then so is L. Since we have the canonical homomorphism $(I + J)/I \cong I/(I \cap J)$, it follows that the sum of all solvable ideals of L is in fact the unique largest solvable ideal of L, which we call its *radical* and denote by Rad L. Now we can give the official definition of semisimplicity: L is semisimple if and only if Rad $L = 0$. Then for any Lie algebra L, we have that $L/\text{Rad } L$ is semisimple (lest L have a solvable ideal larger than Rad L). Note that it is *not* true that if I is a nilpotent ideal of a Lie algebra L such that L/I is also nilpotent, then L is nilpotent (as the example $L = \mathfrak{t}(n)$ makes clear), but it *is* true that if the quotient L/Z is nilpotent, then so is L, where Z is the center of L, for then we have $L^n \subset Z$ for some n, whence $L^{n+1} = 0$.

We conclude by mentioning that the sum of two nilpotent ideals I, J of any Lie algebra L is easily seen to be nilpotent, whence L has a unique largest nilpotent ideal N (in addition to its unique largest solvable ideal Rad L).