Lecture 10-5

We begin with the last family of classical Lie algebras, namely the algebras $L = \mathfrak{so}(2r)$ of type D_r Here the matrices take the same form as in type B_r, except that the first row and column are omitted, so that the matrices have $2r$ rows and $2r$ columns and consist of four $r \times r$ blocks m, n, p, q with $q = -m^T, n^T = -n, p^T = -p$. The subspace D of diagonal matrices consists of all sums $\sum_{i=1}^r d_i(e_{ii} + e_{r+i,r+i})$ The sums of matrix units that are common eigenvectors of the commutation action of the matrices in D are the same as in type B_r, shifting all indices down by 1; the linear functions arising on D are now the ones of the form $\pm(E_i + E_j), \pm(E_i - E_j)$ for $1 \leq i < j \leq r$. The dimension of L is $2r^2 - r$.

We have now already met all but finitely many of the basic objects of study in the course! We now construct an important group of automorphisms attached to any real or complex Lie algebra L (or more generally any Lie algebra L over a field K of characteristic 0), called its *adjoint group*. Let d be a nilpotent derivation of L, so that $d[x,y] = [dx,y] + [x,dy]$ for $x,y \in L$ and $d^n = 0$ for some n. Define the exponential e^d of d by the usual power series $\sum i = 0^n d^i/i!$, taking d^0 as usual to be the identity map; this makes sense for any field K of characteristic 0 since it has only finitely many terms. By the definition of derivation and the binomial theorem we compute that $[e^d x, e^d y] = e^d[x,y]$ for all $x,y \in L$; moreover e^d is invertible on L since its inverse is e^{-d}. In particular if $x \in L$ is such that ad x, the linear map of bracketing with x, is nilpotent, then the exponential e^x of ad x is an automorphism of L. The group generated by all such automorphisms e^x is by definition Int L, the adjoint group of L. If the basefield K is $\mathbb{R}$ or $\mathbb{C}$, then e^x makes sense as an automorphism of L for any x, not just one with ad x nilpotent; in these cases we take Int L to be generated by all such e^x. (Then Int L has the structure of a Lie group, whose Lie algebra is the quotient L/Z of L by its centerI $Z = Z(L)$, defined below.) If L is complex *and semisimple* then the two definitions of Int L (one using all e^x, the other using e^x only for ad x nilpotent) coincide.

We now introduce some important ideals of any Lie algebra L. Its *derived algebra* $[LL]$ is spanned (by definition) over the basefield K by all $[x,y]$ as x,y range over L (it is not true in general that the set of all brackets $[xy]$ is closed under addition). More generally, if I,J are ideals of L, then the Jacobi identity shows that $[IJ]$, spanned by all brackets $[xy]$ as x runs over I and y over J, is an ideal. Another ideal of L is its *center* $Z(L)$, consisting of all $x \in L$ such that $[xy] = 0$ for all $y \in L$. We say that L is *simple* if its dimension is larger than 1 and L has no ideals apart from itself and 0 (any 1-dimensional Lie algebra is necessarily abelian and satisfies this condition, but we do not want to call it simple). We say (for now) that L is *semisimple* if it is a finite direct sum $\oplus_i L_i$ of simple ideals L_i, so that L is the direct sum of the L_i as a vector space, each L_i is simple as a Lie algebra, and $[L_i L_j] = 0$ if $i \neq j$; this is not the official definition but will later be shown to be equivalent to it. In particular, if L is simple or semisimple, we must have $L = [LL], Z(L) = 0$. Whenever $Z(L) = 0, L$ is isomorphic to a linear Lie algebra, that is, to a Lie algebra of matrices. To see this, note that the map from L to $\mathfrak{gl}(L) \cong M_n(K)$ (i.e. to the set of linear maps from the K-vector space L to itself) sending x to ad x is a Lie algebra homomorphism, by the Jacobi identity; its kernel is clearly $Z(L)$, so L is isomorphic to a subalgebra of $M_n(K)$ if $Z(L) = 0$. This homomorphism from L to $M_n(K)$ is called the *adjoint representation* of L; more generally, any Lie algebra homomorphism from L to some $M_m(K)$ is called a representation, the definition paralleling that of a representation of a group. The corresponding notion of *L-module* is a finite-dimensional K-vector space M such that for every $x \in L, m \in M$ there is $x.m \in M$ such that $(x+y).m = x.m + y.m, x.(m_1 + m_2) = x.m_1 + x.m_2, kx.m = x.km = k(x.m)$ if $k \in K$, and finally $x.y.m - y.x.m = [xy].m$ for $x, y \in L, m \in M$. Thus L itself is always an L-module.

As a challenging exercise, try to prove directly that $L = \mathfrak{sl}(n)$ is a simple Lie algebra for $n > 1$, over any field K of characteristic 0. The case $n = 2$, where L has basis h, x, y, where $h = e_{11} - e_{22}, x = e_{12}, y = e_{21}$, is worked out in the text in §2.1; here $[hx] = 2x$, $[hy] = -2y$, and $[xy] = h$. In general one shows that every nonzero ideal of $L = \mathfrak{sl}(n)$ contains first one matrix unit e_{ij} for $i \neq j$, then all such units, and finally all of L. The algebra $L = \mathfrak{gl}(n)$ of all $n \times n$ matrices over K is almost simple, but not quite; it has the ideals $\mathfrak{sl}(n)$ and $K = Z(\mathfrak{gl}(n))$, the latter consisting of all the scalar matrices.

While L has no proper ideals, certain subalgebras of it have many such ideals. Specifically, the subalgebra $\mathfrak{t}(n)$ of upper triangular matrices has the ideal $\mathfrak{u}(n) = [\mathfrak{t}(n), \mathfrak{t}(n)]$ of strictly upper triangular matrices (with zeroes on the diagonal) as an ideal; in fact, note that $\mathfrak{t}(n)$ is also an associative K-algebra under multiplication and $\mathfrak{u}(n)$ is a two-sided associative ideal of it. The quotient $\mathfrak{t}(n)/\mathfrak{u}(n)$, either of Lie algebras or of associative algebras, is naturally isomorphic to the (Lie or associative) algebra $D = \mathfrak{d}(n)$ of diagonal matrices, and D is abelian whether it is regarded as a Lie or associative algebra. Passing to the derived subalgebra $[\mathfrak{u}(n), \mathfrak{u}(n)]$ of $\mathfrak{u}(n)$, we find that it consists of all strictly upper triangular matrices whose superdiagonals (immediately above the main diagonal) are also 0; the quotient of $\mathfrak{u}(n)$ by its derived algebra is again abelian. Iterating the derived subalgebra construction, we find that we eventually get the 0 algebra.

More generally, given an arbitrary Lie algebra L, we define its *derived series* $L^{(0)}, L^{(1)}, \ldots$ via $L^{(0)} = L, L^{(n)} = [LL^{(n-1)}$; if we then have $L^{(n)} = 0$ for some n, then we call L *solvable*. This notion is motivated by and is strongly analogous to a parallel notion for groups which was defined much earlier, in fact by Galois. There are also parallel notions of something called *nilpotence* for both Lie algebras and groups, but in that case the notion for Lie algebras was defined first. We define the *lower central series* of an arbitrary Lie algebra L by $L^0 = L, L^n = [LL^{n-1}]$; we say that L is nilpotent if $L^n = 0$ for some n. Equivalently, L is nilpotent if and only if for some n all n-fold brackets $[\ldots[x_1 x_2]x_3]\ldots x_n]$ are identically 0 in L. Clearly any nilpotent Lie algebra is solvable, but the converse fails, since $\mathfrak{t}(n)$ is solvable but not nilpotent. (The notions of solvability and nilpotence for groups G can be defined in the same way, defining $[GG]$ as the commutator subgroup of G, and then defining the derived and central series for G in the same way as for L.)