Lecture 10-2

Continuing from last time, we now study the classical algebras (those belonging to one of the series A_n, B_n, C_n, or D_n), in more detail. Recall that the *matrix unit* e_{ij} has 1 as its ij-entry while all other entries are 0. There is a simple rule for commutating two such units: $[e_{ij}e_{k\ell}] = \delta_{jk}e_{i\ell} - \delta_{\ell}ie_{kj}$, where δ_{ij} is the Kronecker delta. In all cases the diagonal matrices will play a crucial role, since diagonalizable linear transformations are the easiest ones to understand and if d is diagonal than ad d will continue to act diagonalizably on the algebra of all matrices.

The simplest case is $L = \mathfrak{sl}(n)$ (type A_{n-1}), the Lie algebra of all $n \times n$ matrices of trace 0. Here we have $[e_{ii}e_{ij}] = e_{ij}$. As mentioned previously, bracketing with a fixed diagonal matrix $d = \sum d_i e_{ii}$ of L defines a diagonalizable linear transformation from L to itself. The eigenvalues of this transformation depend on d in a linear way. More precisely, if d is as above, then $[de_{ij}] = (d_i - d_j)e_{ij}$. If we define a linear function E_i on the vector space D of diagonal matrices via $E_i(d) = d_i$, then we can rewrite our last equation as $[de_{ij}] = (E_i - E_j)(d)e_{ij}$. Here we must have $\sum_i d_i = 0$, since we are only looking at matrices with trace 0. Thus the dimension of our subspace D is $n-1$ rather than n; this is why we say that L is of type A_{n-1} rather than A_n. This subspace D is a subalgebra of L, in fact an *abelian* one where all brackets are 0. (This will also be true of the subspace D of diagonal matrices in all the other classical algebras; in general the subscript n in the label X_n of a classical Lie algebra refers to the dimension of the subspace D in it). Here L is spanned by D together with all matrix units e_{ij} with $i \neq j$. The dimension of L is thus $n^2 - 1$.

In all the other families of examples, there is a matrix M such that the Lie algebra consists of all matrices X that are skew-adjoint with respect to the from $(\cdot, \cdot)$ given by $(v, w) = v^t M w$, where the superscript t as usual denotes transpose, so that v^t is a row vector while w is a column one. Skew-adjointness with respect to this form translates to the condition on X that $MX = -X^t M$. We now consider each of the cases in turn.

Starting with $L = \mathfrak{sp}(2r)$ (type C_r), the condition $MX = -X^tM$ says that the upper left $r \times r$ block m of X should be the negative transpose $-q^T$ of the lower right block q, while the upper right and lower left blocks n, p should equal their own transposes. Hence a typical diagonal matrix d in L takes the form $d = \sum_{i=1}^{2r} d_i e_{ii}$, where $d_{r+i} = -d_i$ for $1 \leq i \leq r$. Letting E_i as above be the linear function sending $\sum_i d_i e_{ii}$ to d_i (for $1 \leq i \leq r$ only), we find that M is spanned by D together with the differences $\ell_{ij} = e_{ij} - e_{r+i,r+j}$ (for $1 \leq i, j \leq r, i \neq j$), the sums $\ell'_{ij} = e_{i,r+j} + e_{j,r+i}$ (for $1 \leq i < j \leq r$), their transposes $\ell''_{ij} = e_{r+j,i} + e_{r+i,j}$, and the units $\ell_i = e_{i,r+i}$ and their transposes $\ell'_i = e_{r+i,i}$. Then $[d\ell_{ij}]$ equals $(E_i + E_j)(d)\ell_{ij}$ if $i < j$ and $(-E_i - E_j)(d)\ell_{ij}$ if $j < i$, while $[d\ell'_{ij}] = (E_i - E_j)(d)\ell'_{ij}, [d\ell''_{ij}] = (E_j - E_i)\ell''_{ij}, [d\ell_i] = 2E_i(d)\ell_i, [d\ell'_i] = -2E_i(d)\ell'_i$. The dimension of L is $r + r^2 - r + 2(r + (1/2)(r^2 - r)) = 2r^2 + r$; the dimension of D is r.

Now consider $L = \mathfrak{so}(2r+1)$ (type B_r). Now the condition $MX = -X^t M$ says that X has 0 as its 11-entry; the remaining blocks of entries b_1, b_2 in its first row are the respective negative transposes of c_2, c_1, the remaining blocks of entries in its first column. The remainder of X consists of four blocks m, n, p, q as in the previous case, but this time with $q = -m^T, n^T = -n, p^T = -p$. Here D consists of all sums $\sum_{i=1}^{2r+1} d_i e_{ii}$ with $d_1 = 0, d_{r+i} = -d_i$ for $2 \leq i \leq r+1$ and L is spanned by D together with all differences $\ell_{ij} = e_{i+1,j+1} - e_{r+j+1,r+i+1}$ (for $1 \leq i, j \leq r, i \neq j$), all differences $\ell'_{ij} = e_{i+1,r+j+1} - e_{j+1,r+i+1}$ (for $1 \leq i < j \leq r$), all differences $\ell''_{ij} = e_{r+i+1,j+1} - e_{r+j+1,i+1}$ (for $1 \leq j < i \leq r$), and all differences $\ell_i = e_{1,r+1+i} - e_{i+1,1}, \ell'_i = e_{1,i+1} - e_{r+1+1,1}$ (for $1 \leq i \leq r$). We have $[d, \ell_{ij}] = (\pm(E_i + E_j)(d)\ell_{ij}$ (according as $i < j$ or $j < i$, as before), while we have $[d, \ell'_{ij}] = (E_i - E_j)(d)\ell'_{ij}, [d\ell''_{ij}] = (E_i - E_j)(d)\ell''_{ij}, [d\ell_i] = E_i(d)\ell_i, [d\ell'_i] = -E_i(d)\ell'_i$, where $E_i(d) = d_{i+1}$ for $1 \leq i \leq r$. Here the dimension of D is again r and the dimension of L is again $2r^2 + r$. Notice that we get the same linear functions arising on D as in type C_r, except that $\pm 2E_i$ is replaced by E_i.