Sample Final Problems–Math 506

1. State the version of the Nullstellensatz that holds for arbitrary basefields.

2. Let $f : A \to B$ be a homomorphism of rings. Show that the induced map $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is continuous with respect to the Zariski topology on both spectra.

3. State the theorem on existence of primary decompositions of ideals in a Noetherian ring, indicating also the two features of this decomposition that are uniquely determined by the ideal.

4. Give an example of a Noetherian domain of dimension one that is not a PID.

5. Give a sufficient condition for a polynomial $q \in \mathbb{Z}[x]$ to have a solution in $\mathbb{Z}_p$, the ring of p-adic integers, for some prime p.

6. Show that $\mathbb{Z}[i]$, the ring of Gaussian integers, is a faithfully flat extension of $\mathbb{Z}$, first by showing that it is flat, and then verifying its faithfulness by showing that the induced map on prime spectra is surjective.

7. Give a necessary and sufficient condition in terms of ideal factorization for one nonzero ideal I in a Dedekind domain A to divide another.

8. Show that there is no surjective morphism from K^m to K^n for any algebraically closed field K if $m < n$.