HW #1, DUE 4-7

MATH 506A

1. Let R be a (commutative) ring such the localization R_M of R at any maximal ideal M is Noetherian and such that there are only finitely many maximal ideals containing any nonzero element of R. Show that R is Noetherian. (Let $I \neq 0$ be an ideal of R and let $M_1, \ldots, M_r$ be the maximal ideals containing it. Choose $x_0 \in I, x_0 \neq 0$ and let $M_1, \ldots, M_{r+s}$ be the maximal ideals containing x_0. For $1 \leq j \leq s$ pick $x_j \in I, x_j \notin M_{r+j}$. The ideal IR_{M_i} is finitely generated for $1 \leq i \leq r$, so we can choose $x_{s+1}, \ldots, x_t \in I$ whose images generate IR_{M_i} for $1 \leq i \leq r$. Let $I_0 = (x_0, \ldots, x_t) \subset R$. Show that I_0 and I generate the same ideal in R_M for all maximal ideals M of R and deduce that $I_0 = I$.)

2. Let $R = K[x_1, \ldots]$ be the polynomial ring in infinitely many variables x_i over a field K. Let $m_1, m_2, \ldots$ be an increasing sequence of integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all i (e.g. $m_i = 2^i$) and let $P_i = (x_{m_i+1}, \ldots, x_{m_{i+1}})$ be the prime ideal of R generated by the given variables. Let S be the complement of the union of the P_i. Show that the localization $S^{-1}R$ satisfies the hypotheses of Problem 1, so that $S^{-1}R$ is Noetherian. Also show that each $S^{-1}P_i$ has height $m_{i+1} - m_i$, so the dimension of the Noetherian ring $S^{-1}R$ is infinite.

3. Let $P_1, \ldots, P_m$ be prime ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ (with K algebraically closed) such that no P_i contains another. The intersection I of the P_i is then a radical ideal. Show that the multiplicity of each P_i in the R-module R/I (as defined last quarter) is one, by setting $I_j = \cap_{i=j}^m P_i$ for $1 \leq j \leq m, I_{m+1} = R$, filtering R/I by the increasing intersections $I_1/I, I_2/I, \ldots, I_{m+1}/I$ and showing that, when localized at P_i, exactly one graded piece I_j/I_{j-1} is isomorphic to R/P_i while the others are 0.

4. Suppose that R is complete with respect to an ideal I and that M is an R-module. Call M *separated* with respect to I if $\cap_n I^n M = 0$. Show that if M is separated and the images of $m_1, \ldots, m_n \in M$ generate M/IM, then the m_i generate M.

5. The *Jacobson radical* J of a commutative ring R is the intersection of all maximal ideals of R. Show that J consists exactly of those $r \in R$ such that $1 + xr$ is a unit for all $x \in R$. Deduce that if R is complete with respect to an ideal I, then (the image of) I lies in the Jacobson radical of R.