HW #3, DUE 4-21

1. Exercise 3.10, Eisenbud, p. 112 (2004 edition)

2. Exercise 3.15, Eisenbud, p. 113 (2004 edition)

3. Prove the Jordan-Holder Theorem, which asserts for a left module M over a ring R (possibly noncommutative) that if M admits two finite filtrations $(M_i), (M_i')$ with $M_0 = 0 \subset \dots M_n = M$, similarly for the M_i', and if the M_i/M^{i-1} and M_j'/M_{j-1}' are irreducible R-modules, then there is a permutation π of the indices so that M_i/M_{i-1} is isomorphic to $M_{\pi(i)'}/M'_{\pi(i)-1}$ for all i. (Argue by induction on the length of the shorter filtration, say (M_i), and show that if j is the least index with $M_1 \cap M_j' \neq 0$, then in fact $M_1 \subset M_j'$ and we can in effect mod out by M_1 in both filtrations.)

4. Give an example of an Artinian $\mathbb{Z}$-module that is not Noetherian.

5. For k a (not necessarily algebraically closed) field, let A be a finitely generated k-algebra. Show that A is Artinian as a ring if and only if it is finite-dimensional over k.