FINAL EXAM

1. Find two examples of a ring R with exactly two prime ideals, one where one of these ideals is contained in the other, the other with neither ideal contained in the other.

The easiest examples are probably the ring $K[[x]]$ of power series in one variable x over a field K in the first case and the ring $\mathbb{Z}/(6)$ of integers mod 6 in the other.

2. Classify as completely as you can the fields that are homomorphic images of $\mathbb{Q}[x, y, z]$, the polynomial ring in three variables over $\mathbb{Q}$.

These are exactly the finite extensions of $\mathbb{Q}$ (since any such extension is in fact simple and so generated by three elements).

3. Let R be a Noetherian ring with nilradical 0. Show that the total ring of quotients $K(R)$ (obtained from R by localizing by all non-zero-divisors) is an Artinian ring.

The set D of zero divisors in any such ring is the union of the prime ideals P_i arising as radicals of its primary components. Since 0 is its own radical, none of these radicals contains another (by the uniqueness of the radicals of the primary components, since the radical of a finite intersection is the intersection of the radicals). Hence any prime ideal, being contained in D by construction, must equal one of the P_i, again by uniqueness of the radicals in any primary decomposition, l

4. Show that 2 is a square in the ring $\mathbb{Z}_7$ of 7-adic integers.

This follows at once from Hensel's Lemma, since $3^2 = 2$ in $\mathbb{Z}/(7)$, and 3 is in fact a simple root of the polynomial $x^2 - 2$ in this ring.

5. If K is an algebraically closed field, classify the subvarieties of K^n whose coordinate rings are Artinian.

These are exactly the finite subsets of K^n, since the coordinate rings in question must be direct products of finitely many copies of K (the only possible Artinian local integral domain over K).

6. Characterize the primary ideals of a Dedekind domain A in terms of the prime ideals of A.

These are exactly the powers of the prime ideals in A.

7. If R is a ring, show that the ring $R[x]$ of polynomials in one variable x over R is a faithfully flat extension of R.

This follows at once since $R[x]$ is free and thus flat over R, while any prime ideal P of R has the proper extension $P[x]$ in $R[x]$, which contracts to P in R.

8. Let k be an algebraically closed field, A an affine domain over k. We have computed the dimension of A in four different ways (three of them involving a choice of maximal ideal M of A, but all in fact giving the same answer for every M). Describe these ways as clearly as you can (but without giving proofs).

The first is the transcendence degree of the quotient field K of A over k. For the other three, one first chooses a maximal ideal M of A. Then one takes either the supremum of the lengths n of all strictly ascending chains of primes $P_0 = 0 \subset \cdots, \subset P_n = M$ (Krull dimension); or the degree d of the polynomial $\dim_k M^n/M^{n+1}$ as a function of n, for n sufficiently large; or the minimum number of generators of any M-primary ideal in A.