LECTURE 5-8

We now prove Serre's Criterion for being a (finite) direct product of normal domains. If R is a direct product $R_1 \times \cdots \times R_n$, then any prime in R takes the form $R_1 \times \cdots \times R_{i-1} \times Q_i \times \cdots \times R_n$ for some prime Q_i of R_i; this ideal is associated to 0 if and only if Q_i is associated to 0 in R_i. The primes associated to a non-zero-divisor $(a_1, \ldots, a_n)$ are those of the above form with Q_i an associated prime of a_i in R_i. Now it follows at once from results proved last time that if each of the R_i is normal, then is satisfies condition $(R1)$ and so does R; likewise it satisfies $(S2)$ since a localization of R at a prime of codimension c coincides with a localization of some R_i at a prime of codimension c. Conversely, if R satisfies both these conditions, then we first show that R is reduced. If $0 = \cap I_j$ is a minimal primary decomposition with I_j a P_j-primary ideal, then $(S2)$ forces R_{P_j} to be a field, since each P_j has codimension 0 by $(R1)$. Hence $I_j = P_j$ and R is reduced. Now we can apply the last result proved last time. Since for each prime P associated to a non-zero-divisor in R the localization R_P is integrally closed, we see that R is integrally closed in its total quotient ring $K(R)$. This last ring is reduced and of dimension 0 whence it is Artinian and must be a direct product of fields $K_j = (R/P_j)_{P_j}$. Letting e_j be the identity element of K_j, we have $e_j^2 = e_j, e_i e_j = 0$ if $i \neq j$. But then each e_j is integral over R, forcing $e_j \in R$, and R is the direct product of the rings $Re_j = R/P_j$. Moreover, integral closure of R in $K(R)$ implies that each R/P_j is integrally closed in K_j, so that R is a finite direct product of normal domains, as claimed.

We now generalize many of the facts we proved about arbitrary ideals in Dedekind domains to ideal of codimension one in Noetherian rings. We begin with a definition that could have been given already when we were working with Dedekind domains, but which was not needed for what we proved at that time. Call a module I of any commutative ring R *invertible* if the localization I_P of I at any prime ideal P of R is isomorphic to R_P; of course it suffices to check this for maximal ideals M, since R_P is just a further localization of R_M. Write I^* for $\hom_R(I, R)$ (by analogy with standard notation used in linear algebra). We will make frequent use of the natural map $\mu : I^* \otimes I \to R$ sending $\rho \otimes a$ to $\rho(a)$. (We will see that any invertible module is isomorphic to an ideal, so our notation is not as misleading as it looks.) Now any principal ideal $I = (x)$ of R generated by a non-zero-divisor x is invertible, for then I_P is free of rank one over the local ring R_P, so is isomorphic to R_P. We have already seen that any nonzero ideal I over a Dedekind domain is invertible, for then R_P is a DVR and thus in particular a PID. Thus we get many examples of nonprincipal invertible ideals.

Our next task is to compare invertible modules up to isomorphism to *fractional ideals*, which are by definition R-submodules of the total quotient ring $K(R)$. If I is a finitely generated fractional ideal, then it is isomorphic to an ordinary principal ideal, as one sees by looking at a common denominator of the generators. If $I \subset K(R)$ is any set, then we write I^{-1} for the set $\{s \in K(R) : sI \subset R\}$. Then our main result states that *if R is Noetherian and I is an R-module, then I is invertible if and only if the map μ above from $I^* \otimes I$ to R is an isomorphism. Every invertible module is isomorphic to a* *fractional ideal of R; every invertible fractional ideal of R contains a non-zero-divisor of R. If I, J are invertible modules, then the natural maps $I \otimes J \to IJ$ taking $s \otimes t$ to st and $I^{-1}J \to \hom_R(I, J)$, taking $t \in I^{-1}J$ to the map sending $a \in I$ to ta are isomorphisms. In particular, $I^{-1} \cong I^*$. Finally, if $I \subset K(R)$ is any R-submodule, then I is invertible if and only if $I^{-1}I = R$.*

To prove this we being by noting that if I is invertible then μ localizes at any prime P to the obvious isomorphism $R_P^* \otimes_{R_P} R_P \cong R_P \otimes_{R_P} R_P \cong R_P$, whence μ itself is an isomorphism. Conversely, suppose μ is an isomorphism and that 1 is the image of $\sum_i \phi_i \otimes a_i$ under it. Then the localization μ_P of μ at P is an isomorphism for every prime P; we will show that $I_P \cong R_P$ and that I_P is generated by a_i for some i. Some $\phi_i(a_i)$ must lie outside P; letting $v = \phi_i(a_i^{-1})$ we see that $a = va_i$ goes to 1 under μ_P. Then $I_P = R_P a \oplus \ker(\phi_I)_P$; similarly $I_P^* = R_P\phi_i \oplus \ker a$ and $R_P\phi_i \cong R_P$. Now $I_P^* \otimes I_P = R_P a \otimes R_P \phi_i \oplus (\ker \phi_i)_P \otimes R_P \phi_i \oplus \cdots$ and $(\ker \phi_i)_P \otimes R_P \phi_i$ maps to $(\phi_i)_P \ker(\phi_i)_P = 0$ under the isomorphism μ, so $(\ker \phi_i)_P = 0$ and $(\phi_i)_P$ is an isomorphism sending a_i to a generator, as claimed. As this holds for all P, we see that I is generated by the $a_1, \ldots, a_n$ as well.

We will prove the other parts next time.