# LECTURE 5-5

One more topic from Chapter 10 of Eisenbud: we call a sequence $x_1, \ldots, x_d$ of elements in a ring $R$ a *regular sequence*, or *$R$-sequence*, if the ideal $(x_1, \ldots, x_d)$ is proper and for each $i$ the image of $x_{i+1}$ in $R/(x_1, \ldots, x_i)$ is a non-zero-divisor. Recall from last quarter that a Noetherian local ring $R$ of dimension $d$ is called *regular* if its maximal ideal is generated by $d$ elements (the minimum possible); then any set of $d$ generators of this ideal is called a *system of parameters*. We saw last quarter that a regular local ring must be an integral domain, since its associated graded ring with respect to the standard filtration by powers of the maximal ideal is a polynomial ring over the quotient by this maximal ideal. Since $R/(x_1, \ldots, x_i)$ is regular local whenever $R$ is and $x_1, \ldots, x_d$ generate the maximal ideal, it follows that *any system of parameters is a regular sequence*; the image of $x_{i+1}$ in $R/(x_1, \ldots, x_i)$ cannot be 0, lest the maximal ideal be generated by fewer than $d$ elements. We will return to regular sequences in Chapter 18.

Following Chapter 11 of Eisenbud, we now treat general Noetherian rings from the viewpoint of our earlier study of Dedekind domains, generalizing a number of results for ideals in a Dedekind domain to primes of codimension one in a Noetherian domain. Our first major result states that *a Noetherian domain $R$ is normal (=integrally closed) if and only if for every prime ideal $P$ associated to a principal ideal $I$ (that is, to the module $R/I$), the localization $P_P = PR_P$ of $P$ in $R_P$ is principal.*

Before starting the proof, we note that the condition that $P_P$ be principal implies that the codimension of $P$ is 1, by the principal ideal theorem, so the given condition implies that every prime associated to a principal ideal has codimension one. Next, if $P$ has codimension 1, so that $R_P$ is one-dimensional, then we have seen last quarter that $P_P$ is principal if and only if $R_P$ is a discrete valuation ring (DVR). Hence our necessary and sufficient condition is equivalent to requiring that every localization of $R$ at a prime of codimension one is a DVR and that every prime associated to a principal ideal has codimension 1. Now we can begin the proof. We first show that the given condition implies that $R$ is normal. Since the intersection of normal domains with the same quotient field is obviously again normal, it is enough to show that $R$ is the intersection of its localizations at primes minimal over principal ideals. We will prove a more general result, valid for reduced rings (with nilradical 0, or equivalently with no nonzero nilpotent elements): *if $R$ is reduced and Noetherian, then an element in its total quotient ring $K(R)$ (obtained from $R$ by localizing by all non-zero-divisors) lies in $R$ if and only if the image of $x$ in $K(R)_P$ lies in $R_P$ for all primes $P$ associated to principal ideals generated by non-zero-divisors.* To see this, let $a/u \in K(R)$ with $a, u \in R$ and $u$ not a zero divisor. If $a/u \notin R$, then $a \notin (u)$, whence the image of $a$ fails to lie in $(u)_Q$ for some prime $Q$ containing $u$ and hence also for some associated prime $Q$ of $R/(u)$ (since $m = 0$ for an element $m$ of an $R$-module $M$ if and only if the image of $m$ in $M_P$ f is 0 for every maximal associated prime of $M$). Now we prove the converse. If $R$ is normal and $P$ is associated to $(a)$ for some principal ideal $(a)$, then $P$ is the annihilator of some $b \in R$ mod $(a)$, with $b \notin (a)$. We must show that $P_P$ is principal. Localizing if necessary, we may assume from the outset that $R$ is local with maximal ideal $P$. Let $K$ be the quotient field of $R$ and set $P^{-1} = \{x \in K : xP \subset R\}$; we generalize this definition to any ideal $I$ of $R$. We clearly have $P \subset P^{-1}P \subset R$, so the only possibilities are $P^{-1}P = P$ or $P^{-1}P = R$ (since $P$ is maximal). The first possibility would imply by the finite generation of $P$ that every element of $P^{-1}$ is integral over $R$, forcing $P^{-1} = R$ by normality; but $Pb \subset (a)$, implying $b/a \in P^{-1} = R, b \in (a)$, a contradiction, so we must have $P^{-1}P = R$. Then $rP$ is not a subset of $P$ for some $r \in P^{-1}$, forcing $rP$ to contain a unit of $R$ and thus $rP = R$. Then $P = Rr^{-1}$ is principal, as desired.

As a consequence every normal Noetherian domain is the intersection of its localizations at codimension-one primes, for we have shown that any reduced ring is the intersection of its localizations at primes associated to (the ideals generated by) non-zero-divisors, and we have just shown that such primes must be principal, forcing them to have codimension one. The geometric version of this corollary states that if $X$ is a normal variety (having normal coordinate ring) and if $Y \subset X$ has codimension at least 2, then any rational function regular on $X - Y$ extends to one regular on $X$. There is an analogous fact in complex variables: if $X$ is a normal analytic variety (in an analogous but different sense from the one above) and if $Y$ is a subset of codimension at least 2, then any meromorphic function on $X$ holomorphic outside $Y$ is in fact holomorphic in $X$.

Serre (a very famous name you will soon get to know if you haven't already) observed that the condition of our main result applies much more generally to rings that are not domains; by modifying them slightly we can distinguish normal rings among Noetherian rings. Here a *normal ring* is a reduced ring integrally closed in its total quotient ring. Then it turns out that a normal ring is a direct product of normal domains, so if the ring is local or graded, it must in fact be a domain already. That is, *a Noetherian ring $R$ is a finite direct product of normal domains if and only if (R1) every associated prime of a principal ideal generated by a non-zero-divisor has codimension one, while every associated prime of 0 has codimension 0; and (S2) every localization of $R$ at a codimension-one prime is a DVR, while every localization of $R$ at a codimension-0 prime is a field.*

The $R$ in the first condition stands for "regular", while the $S$ in the second one stands for Serre. We will give the proof next time.