LECTURE 5-31

We now review the four ways that we have discussed to measure the size of an affine domain R over a field k. The last one we introduced is the easiest to define; it is just the transcendence degree of k of the quotient field K of R. We defined the dimension of R in this way only quite recently, but we could have done it last quarter, as we saw then that any such R (indeed any affine ring, domain or not) is a finite integral extension of some polynomial ring $k[t_1, \ldots, t_d]$; being thus not much larger than $k[t_1, \ldots, t_d]$ itself, it is natural to take its dimension to be d, and this is indeed the transcendence degree of its quotient field if R is a domain (though if R is not a domain, we have to be a bit careful, as there is no single field canonically attached to R, and its variety might have different components of different dimension).

The other three measures of dimension all involve a choice of maximal ideal M of R, though all give the same answer for every M, at least for domains R. The first of these (Krull dimension) is mentioned already in Dummit and Foote; it is the supremum of the lengths n of all strictly increasing chains of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n = M$ in R ending at M. It is not at all obvious a priori that this supremum is finite, and in fact it need not be if one looks at all chains not ending in a *fixed* maximal ideal, but in our situation the supremum is indeed finite.

Next we have the rate of growth of powers of M, more precisely the dimension of the quotient M^n/M^{n+1} (or more generally the quotient of two successive terms of a stable M-filtration of R) as a vector space over the field R/M. This dimension is a polynomial in n for sufficiently large n, whose degree is the dimension of R. The leading term of this polynomial gives more refined information about the rate of growth of powers of M, which we have used to define the notion of multiplicity (replacing M by a finitely generated R-module N and using a finite filtration of N whose graded pieces take the form R/P for various prime ideals P of R; the multiplicity counts the number of times R/P occurs in N).

Finally we have the minimal number of generators required for any ideal containing a power M^n of M, or equivalently any ideal with radical M. We showed that any prime ideal of codimension d is minimal over some ideal $(x_1, \ldots, x_d)$ with the $x_i \in P$; if we allow one more generator x_{d+1} we can guarantee that P is the only minimal prime over $(x_1, \ldots, x_{d+1})$, or equivalently the radical $\sqrt{(x_1, \ldots, x_{d+1})} = P$. Whether one can replace these last $d+1$ generators with d generators (of P up to radical) is one of the great open questions in algebraic geometry, for homogeneous prime ideals P of polynomial rings of codimension one over algebraically closed fields k. In particular the dimension of M/M^2 over R/M (where M is maximal in a Noetherian ring R) is always at least the codimension of M; if equality holds, R is said to be *nonsingular at* M. If R is affine over k and nonsingular at some maximal ideal M with $A/M \cong k$, then the associated graded ring $G_M(R)$ is just the polynomial ring in $\dim R$ variables over k; so in some sense every Noetherian ring R nonsingular at a maximal ideal M looks like a polynomial ring over R/M to a first approximation.

We have shown in class that a ring is Artinian if and only if it is Noetherian and has dimension 0; any Artinian ring is a finite direct product of Artinian local rings, in each of which the maximal ideal is nilpotent. An Artinian affine ring is necessarily finite-dimensional (not just finitely generated) over its basefield k. The next simplest situation occurs when R is a Noetherian domain of dimension one, so that every nonzero prime ideal is maximal. To get a really nice theory of such rings one needs a further assumption on R; usually this is taken to be that R is integrally closed in its quotient field. Then every nonzero ideal of R is uniquely a product of prime ideals, every primary ideal of R is a power of prime ideal (and conversely every power of a prime ideal is primary), and every localization R_P of R at a prime ideal P is a DVR (a PID with only one nonzero prime ideal). Nonzero ideals of R form a group under multiplication if one mods out by the principal ideals; this is called the *class group* of R and in general can be any abelian group, but in the special case where R is the integral closure of $\mathbb{Z}$ in a finite extension K of $\mathbb{Q}$ this group is finite. It may be identified with the *Picard group* of isomorphism classes of invertible modules that we studied this quarter (for Dedekind domains, but not in general).

We will wrap up by a saying a few words about complete rings next time.