LECTURE 5-26

In this last lecture on new material we sketch a few results on regular sequences and Cohen-Macaulay rings (Chapters 17 and 18 of Eisenbud); the latter are a very important class of rings capturing and generalizing the key nice properties of affine rings that we observed in Chapter 13. We have already defined the notion of *regular sequence* in Chapter 10: given a proper ideal I of R, a sequence $x_1, \ldots, x_d$ of elements in I is regular if x_i is a non-zero-divisor in $R/(x_1, \ldots, x_{i-1}$ for all i. The *depth* of I, denoted depth(I), is the length of the longest possible regular sequence. A basic result proved in Chapter 17, using the Koszul complex, is that the depth of every ideal I of a Noetherian ring R is finite and in fact bounded by the codimension of I; moreover, any two maximal regular sequences in I have the same length. One should think of the depth as an arithmetic measure of the size of I, while the codimension is a geometric measure of size. Like the codimension of I, its depth turns out to depend only on the radical of I; again like the codimension, the depth of an ideal I generated by r elements is at most r. More generally, if M is an R-module and I an ideal with $IM \neq M$, then a *regular M-sequence in I* is a sequence $x_1, x_2, \ldots$ of elements of I such that x_i is a not a zero divisor on $M/(x_1, \ldots, x_{i-1})M$ for any i; then any two maximal M-sequences in I have the same length, denoted depth$(I.M)$. For $M = R$ this reduces to depth(I) as defined above. If $IM = M$, then by convention depth$(I, M) = \infty$. Generalizing the above inequality for depth(I), we find that depth(I, M) is bounded above by the length of any maximal chain of prime ideals descending from a prime ideal containing I to an associated prime of M. The proof, as for codimension, is inductive, using a crucial lemma that if R is local with maximal ideal P, M is finitely generated over R, I is an ideal of R, and $y \in P$, then depth$((I, y), M)$ is at most depth(I, M) plus one. This depth can be measured in cohomological terms, as the smallest degree for which the so-called *Koszul complex* attached to I and M has nonvanishing cohomology.

A ring R is said to be *Cohen-Macaulay* if the depth of any maximal ideal P equals its codimension; if so, then any proper ideal I of R has the same depth and codimension. This property localizes in a nice way: *R is Cohen-Macaulay if and only if R_P is for every maximal ideal P, or if and only if R_Q is for every prime ideal Q.* Also *a local ring is Cohen-Macaulay if and only if its completion is.* Then a key result states that *Cohen-Macaulay rings are universally catenary; moreover, in a local Cohan-Macaulay ring R any two maximal chains of prime ideals have the same length and every associated prime of R is minimal.* Moreover, a local Cohen-Macaulay ring is *equidimensional* in the sense that all maximal ideals have the same codimension and all minimal primes have the same dimension. A closely related result first proved by Macaulay for polynomial rings is the Unmixedness Theorem, stating that *if an ideal I in a Noetherian ring R is generated by n elements and has codimension n, then all minimal primes over that ideal have codimension n. If moreover R is Cohen-Macaulay then every associated prime of I is minimal over I.* A nice example comes from a situation arising earlier. Let R be a Cohen-Macaulay ring (e.g. a field) and S the polynomial ring in pq variables over R (known to be again Cohen-Macaulay). Label the variables as x_{ij} with $1 \leq i \leq p, 1 \leq j \leq q$ and set up a $p \times q$ *generic* *matrix* M whose ij-th entry is the variable x_{ij}. Then the quotient of S by the ideal generated by the $r \times r$ minors of M is Cohen-Macaulay, for any r at most the minimum of p and q.

We now start reviewing the course (a little early, since there is no class on Monday). You are responsible only for the material in Atiyah-Macdonald, including the Dedekind domain results that were already covered last quarter. We begin with the material on primary decomposition and associated prime ideals. For simplicity, we focus here on ideals, which already capture all features of the more general case of modules. Any ideal I in a Noetherian ring R (and some other rings) admits a *primary decomposition*, realizing it as a finite intersection of *primary ideals*, which are ideals J such that whenever a product xy lies in J, then either $x \in J$ or $y^n \in J$ for some n; equivalently, every zero divisor in R/J is nilpotent. We normalize our decompositions $\cap_I Q_i$ so that the Q_i have distinct radicals and no Q_i contains the intersection of the others. Any primary ideal Q has a prime radical P; we say that Q is P-primary. Then the *radicals* of the primary ideals occurring in any decomposition of a fixed ideal I depend only on I, as do the primary ideals themselves corresponding to primes in this set of radicals not containing others (isolated primes). We know already from last quarter that there are only finitely many minimal primes containing an ideal I; all isolated primes attached to I occur in this list. The union of their images in the quotient ring R/I coincides with the set of zero divisors in this ring; in particular *the set of zero divisors in a Noetherian ring is the union of finitely many minimal primes*.