LECTURE 5-22

We conclude the course by sketching some of the main ideas in Chapters 14, 15, and 18 of Eisenbud. In Chapter 14, we return to the setting of Chapter 6, looking at a ring extension $R \subset S$ for which S is finitely generated as an algebra over R. We have already seen that the fibers of this extension behave well if S is flat over R; now we will show that if R and S are domains, then S is always flat over a large open subset of R, so that "most" fibers behave well. We begin with the central result in what is known as elimination theory: *If X is a variety over an algebraically closed field k, and if Y is a Zariski-closed subset of $X \times \mathbb{P}^n$, then the image of Y under projection to X is closed.* One should think of this result as saying that the homogeneous variables in $\mathbb{P}^n$ can be "eliminated" to define the projection.

To prove this we reduce immediately to the case $X = K^m$, for in general a closed subset of X is simply the intersection of X and a closed subset of the affine space K^m in which X lives. Then Y is defined by a finite collection of polynomial equations $f_I(x_1, \ldots, x_m, y_0, \ldots, y_n) = 0$ that are homogeneous in the y_i. Now the fiber of projection to X over $\mathbf{a} \in K^m$ is empty if and only if the polynomials $f_i(\mathbf{a}, y_0, \ldots, y_n)$ generate an ideal I containing a power of the irrelevant ideal $J = (y_0, \ldots, y_n)$ in $K[y_1, \ldots, y_n]$, by the homogeneous Nullstellensatz. Let $X_d \subset K^m$ be the subset of points $\mathbf{a}$ such that I does not contain M^d, so that the image of Y is $\cap_d X_d$. For fixed $\mathbf{a} \in K^m$, the polynomial $f_i(\mathbf{a}, y_0, \ldots, y_n)$ is homogeneous, say of degree d_i, and we get all homogeneous polynomials of degree d in I by multiplying the f_i by all monomials of degree $d - d_i$ and taking the span of the resulting set of polynomials. We can thus set up a $p \times q$ matrix M whose entries are the coefficients of the q monomials in the y_i of degree d in the p polynomials obtained from all the f_i by multiplying by all monomials of degree $d - d_i$; these entries are polynomials in the coordinates of $\mathbf{a}$. Then X_d consists exactly of those points for which the rank of M is less than q, i.e. for which all $q \times q$ minors of M vanish. This is a closed condition, so we get a set of polynomials equations defining the image of Y, as desired.

This result fails if $\mathbb{P}^n$ is replaced by K^n, as the simple example of the subvariety of K^2 defined by the equation $xy = 1$ shows. You will prove a related result in homework for next week, that if X is an irreducible variety and Y a closed subset of $X \times \mathbb{P}^n$ such that the fibers of Y under projection to the first factor are irreducible and of constant dimension, then Y is irreducible; again this result fails if either $\mathbb{P}^n$ is replaced by K^n or the hypothesis that the fibers have constant dimension is dropped. As a simple consequence *the image of a projective variety under a morphism is closed; more precisely, if Y is projective over a field k and $\pi : Y \to X$ is a k-morphism to a projective variety X, then $\pi(Y)$ is Zariski-closed in X.* As a typical application of this consequence, we show that *over an algebraically closed field k, the condition on a polynomial of fixed degree d in n variables that it can be factored nontrivially is equivalent to the vanishing of certain polynomials in its coefficients.* To see this, let V_d be the projective space of lines in the vector space of polynomials of degree at most d over k. Since a polynomial is irreducible if and only if any nonzero scalar multiple of it is irreducible, then it makes sense to speak of the reducibility or irreducibility of any element of V_d. If $d = e + f$, then the multiplication map induces a morphism $V_e \times V_f \to V_d$ whose image is the set of polynomials with factors of degrees at most e and at most f. Since $V_e \times V_f$ is a projective variety, its image is closed, and the set of reducible polynomials is a finite union of such images. Similarly, one shows that the sets of polynomials that are perfect dth powers or have dth powers as factors are closed.

Our basic result on generic flatness is in fact a generic freeness result: *let R be a Noetherian domain, S a finitely generated R-algebra. If M is a finitely generated S-module, then there is $a \in R, a \neq 0$, such that the localization $M[a^{-1}]$ of M by (powers of) a is free over $R[a^{-1}]$. If in addition $S = \oplus_i S_i$ is graded, with R lying in degree 0, and if M is graded as an S-module, then a may be chosen so that each graded component of $M[a^{-1}]$ is free over R.*

We prove this by induction on dimension d of $K \otimes_R S$, using a by now classical technique known in French as dévissage and in English as unscrewing. The base case is the one where $K \otimes_R S = 0$, which we regard as having dimension -1. In this case there is $a \in R, a \neq 0$ annihilating M, whence $M[a^{-1}] = 0$, as required. In general, by Noether normalization, we know that there are K-algebraically independent $x_1, \ldots, x_d$ in $K \otimes S$ such that $K \otimes S$ is finitely generated as a module over $K[x_1, \ldots, x_d]$. If S is graded as above, then a simple modification of the proof of Noether normalization shows that we may take the x_i to be homogeneous; multiplying by suitable elements of R we take the x_i to lie in S in both cases. Now let $b_1, \ldots, b_t$ generate S as an R-algebra. Each b_i satisfies an integral equation over $K[x_1, \ldots, x_d]$. Clearing denominators, we may write this as a polynomial equation with coefficients in R and leading coefficient say c_i. Let b be the product of the c_i. Then $S[b^{-1}]$ is integral and thus finite over $S' = R[b^{-1}][x_1, \ldots, x_d]$ and $M' = M[b^{-1}]$ is finite over S' as well. Now we saw last quarter that there is a finite filtration of M' by S'-submodules with graded pieces isomorphic to quotients of S'/Q_i by prime ideals Q_i. If $Q_i \neq 0$, then the dimension of $K \otimes_R S'/Q_i$ is less than d, so by inductive hypothesis there is a_i such that $S'/Q_i[a_i^{-1}]$ is free over $R[a_i^{-1}]$. If $Q_i = 0$, then S' is already free over R (with basis the monomials in the x_i). Taking the product a of b and the a_i, we see that $M[a^{-1}]$ has a finite filtration over $R[a^{-1}]$ with free graded pieces and so is free, as desired. If S and M are graded as above, then the Q_i may be taken homogeneous, in which case the homogeneous components of $S'/Q_i[a^{-1}]$ are free over R, and the remaining assertion follows.

As a consequence we get the *upper semicontinuity of fiber dimension*: if X is a variety over an algebraically closed field K and Y is Zariski-closed in $X \times \mathbb{P}^n$, then for any number e, the set X_e of points $p \in X$ such that the fiber of Y over p has dimension at least e is closed in X.