LECTURE 5-15

Continuing with Chapter 11, we begin with the elementary observation that if $\dim R = 1$, then for any non-zero-divisor $a \in R$ the quotient ring $R/(a)$ has dimension 0 and thus finite length. Hence we have a map $a \to \ell(a)$ sending a non-zero-divisor $a \in R$ to the length $\ell(a)$ of the quotient $R/(a)$ as an R-module. We will soon see that this extends to a homomorphism from $K(R)^*$ to $\mathbb{Z}$ (but it is *not* in general a valuation). Our main result is the following: *for any Noetherian ring R there is a map $\phi : C(R) \to \mathrm{Div}(R)$ sending an invertible ideal I of R to*

$$\phi(I) = \sum_P \ell(R_P/I_P)P$$

where the sum runs over all codimension-one primes of R containing I, but has only finitely many nonzero terms. If $\dim R = 1$, then there is a map $C(R) \to \mathbb{Z}$ sending an invertible ideal I to $\ell(R/I)$

We begin with the general remark that given abelian groups G, H and a subset S of G that generates G, to define a homomorphism π from G to HH it is enough to define π on S in such a way that the products of its images of the elements of any two finite subsets of S with the same product in G are the same (one checks immediately that π extends uniquely to G and respects both products and inverses there). In the current situation we know that the set S of invertible ideals in R generates $C(R)$, so it is enough to show that the recipe above defines $\phi(I)$ for any $I \in S$ in such a way that it respects products. Let $I \in S$ and let P be a codimension-one prime in R. The localization R_P/I_P is one-dimensional and I_P contains a non-zero-divisor, so R_P/I_P is 0-dimensional and indeed has a finite length $\ell(R_P/I_P)$. If $I \not\subset P$, then $\ell(R_P/I_P) = 0$; if $I \subset P$, then P must be one of the finitely many minimal primes over I, so the sum is indeed finite. To show that ϕ respects products, suppose that $I = \prod_j I_j$ with the I_j invertible ideals in R. We must show that the for every codimension-one prime P we have $\ell(R_P/I_P) = \sum_j \ell(R_P/(I_j)_P$. To simplify the notation we may assume that R itself is local and one-dimensional. Then each I_j becomes principal, generated by a non-zero-divisor $a_j \in R$. We have the filtration $R \supset (a_1) \supset (a_1 a_2) \supset \cdots \supset (\prod_j a_j)$; since each a_i is a non-zero-divisor, it is easy to check that the successive quotients are isomorphic to the quotients $R/(a_i)$, so the lengths of these add as required. This proves the first statement. To prove the second, we observe that any R-module of finite length (as any quotient R/I does with I nonzero) admits a finite filtration with graded pieces all of the form R/M_i for some maximal ideal M_i. Last quarter, we saw that the number of times that R/M appears for any fixed M is independent of the filtration, is 0 for all but finitely many M, and if summed for all M as in the right-hand side, gives the length of the module, as required.

If $a \in K(R)^*$ then we call the image $\phi(a)$ of a under ϕ *principal* (as a divisor); the group of divisors modulo principal divisors is called the *codimension-one Chow group* of R and is denoted $\mathrm{Chow}(R)$. (More generally, the quotient of the free abelian group on the codimension-i primes by the subgroup generated by principal divisors modulo codimension-$i+1$ primes is called the ith Chow group of R). Thus we have a map $\psi : \mathrm{Pic}(R) \to$

Chow(R). This map is an isomorphism whenever R is locally factorial (i.e. its localizations are UFDs), but in general ψ need not be either injective or surjective. If R is a normal Noetherian ring, the maps $\phi : C(R) \to \mathrm{Div}(R)$ and ψ :Pic(R) $\to$ Chow(R) are injective, by a simple application of the snake lemma in homological algebra combined with the total order on ideals in a DVR. Probably the simplest example where ψ is not injective occurs when R is an old friend we have visited repeatedly this year, namely the coordinate ring of the variety defined by the equation $y^2 - x^3 = 0$ (see Exercise 11.17, on the homework for next week; here the problem as usual is that R is not normal so not a Dedekind domain.

We conclude with a lemma about torsion-free modules M over one-dimensional Noetherian domains R. Any such module M has a *rank* $r(M)$, defined to be the dimension of $M \otimes_R K$ over the quotient field K of R, or equivalently the maximum number of R-linearly independent elements of M. The lemma states that $\ell(M/xM) \leq r(M)\ell(R/(x))$ *for any* $x \in R, x \neq 0$; we will prove it next time.