LECTURE 5-12

Continuing in Chapter 11 of Eisenbud, we prove further results about ideals of codimension one in Noetherian rings R. Say that an ideal I has *pure codimension one* if every associated prime ideal of I (that is, of the quotient ring R/I as an R-module) has codimension one; the term *unmixed* is often used instead of pure. The trivial case $I = R$ is included in the definition. Then *if R is a Noetherian domain such that for every maximal ideal P the ring R_P is a UFD and if $I \subset R$ is an ideal, then I is invertible (as a module or an ideal) if and only if it has pure codimension one. If $I \subset K(R)$ is an invertible fractional ideal, then I is uniquely expressible as a finite product of powers of prime ideals of codimension one. Thus $C(R)$ is a free abelian group generated by the codimension one primes of R.*

To prove this suppose first that I is invertible. If we localize at any maximal ideal then I becomes principal, generated by a non-zero-divisor. Since a UFD is integrally closed, an earlier theorem shows that I is pure of codimension one. Conversely, if P is a prime ideal of codimension one and M is maximal, then either $P \subset M$, in which case $P_M \subset R_M$ is principal since R_M is a UFD, or $P \not\subset M$, in which case $P_M = R_M$; in both case $P_M \cong R_M$, so P is invertible.

Now we show that any ideal I of pure codimension one is a finite product of codimension one primes. Since a product of invertible ideals is invertible, this will show that I is invertible as well. Suppose for a contradiction that some ideals of pure codimension one do not have this property and let I be maximal among such. Since $I = R$ may be viewed as the empty product (by convention), we may assume that $I \neq R$. Let P be a codimension one prime containing I. Since P is invertible we have $P^{-1}P = R$ and thus P^{-1} properly contains R. If $P^{-1}I = I$, then P^{-1} would consist of elements integral over R; but since R_P is a UFD it is normal, forcing $P^{-1} \subset R$, which is false. Hence $P^{-1}I$ properly contains I and it is a finite product of primes, forcing $I = PP^{-1}I$ to be such a product as well, as desired. By our previous result every invertible fractional ideal may be expressed as a product of powers of codimension one prime ideals (allowing negative powers). It remains to show that two expressions of the same ideal as products of powers of codimension one primes agree up to reordering the factors. Multiplying the expressions by a suitable product of positive powers of primes , we may assume that all powers in both of them are nonnegative. We argue by induction on the sum of the powers in say the first expression. If it is 0, then the first product equals R and the result is immediate. If it is at least 1, then the first product lies in one term Q_1 appearing in the second product, forcing some prime ideal to occur in both expressions (since all prime ideals in sight have codimension one). Cancelling by this prime ideal (by multiplying both sides by its inverse), we reduce this sum and complete the proof.

In particular (as we saw already last quarter) isomorphism classes of nonzero ideals in a Dedekind domain R (or equivalently nonzero ideals modulo principal ideals) form a group under composition, which before we called (and can still call in this setting) the class group, but which now is also the Picard group of R. This group is finite if R is the integral closure of $\mathbb{Z}$ in an algebraic number field (a finite extension of $\mathbb{Q}$); we will not prove this, as to do so requires techniques special to algebraic number theory that do not pertain to commutative algebra. If R is instead the coordinate ring of a smooth curve over an algebraically closed field, then $\mathrm{Pic}(R)$ is finite if and only if R if the curve is rational (birational to $\mathbb{P}^1$). Otherwise the curve will have positive genus g and the Picard group will involve $2g$ copies of the complex torus (=unit circle in the complex plane). In general, the Picard group of a Dedekind domain can be any abelian group A.

The story is very different even for Noetherian domains of dimension one that are not Dedekind domains. For example, as you will see in later homework, the PIcard groups of the curves defined by the equations $y^2 = x^3$ and $y^2 = x^2(x+1)$ over an algebraically closed field K are K (as an additive group) and K^* (as a multiplicative group), respectively. In the first case, the principal ideal (x) of the coordinate ring is not a product of prime ideals; indeed, the only prime containing it is (x, y), and it is not a power of this ideal.

We now refine and extend the familiar notion of divisor in a commutative ring to ideals. Clearly an element b divides another one a if and only if $a \in (b)$; thus an ideal may be regarded as something by which an element might be divisible. Since nonzero ideals in a Dedekind domain are uniquely products of prime ideals, they correspond to finite sets of prime ideals, each occurring with some multiplicity. Accordingly (but more generally) we now call an element of the free abelian group generated by the codimension one prime ideals in a ring R a *divisor*, or more precisely a *Weil divisor*, of R; this group is denoted $\mathrm{Div}(R)$. The elements of the previously described group $C(R)$ of invertible ideals of R will now be called *Cartier divisors*. In general these two sets are very different, but we will see next time that there is a natural homomorphism from $C(R)$ to $\mathrm{Div}(R)$.