LECTURE 5-10

We continue with proving the result stated last time. Let I be an invertible ideal; we must show that it embeds into $K(R)$. We know that $K(R)$, as an Artinian ring, is semilocal; that is, it has only finitely many maximal ideals, these all taking the form $PK(R)$, where P is a maximal associated prime of R. For every such P, we have $I \otimes K(R) = I_P \cong R_P \cong K(R)_{PK(R)}$. By Exercise 4.13 (to be included in next week's HW) this forces $I \otimes K(R) \cong K(R)$. Next we show that the localization map $\phi : I \to K(R) \otimes I = I_U, U$ the set of non-zero-divisors in R, is injective; to do this it is enough to show that it is injective when localized at a maximal ideal P. The map ϕ_P is the localization map sending I_P isomorphically onto R_P and then to $K(R) \otimes R_P = (R_P)_U$; as the element of U are not zero divisors, ϕ_P is injective, as required. The map $I \to K(R) \otimes I \cong K(R)$ is the desired embedding; we have already seen that I is finitely generated. Now suppose that $I \subset K(R)$ is a finitely generated fractional ideal such that $I \cap R$ consists of zero divisors. Because I is finitely generated, there is a non-zero-divisor $u \in R$ such that $uI \subset R \cap I \subset R \subset K(R)$. Since $I \cap R$ consists of zero divisors, there is a nonzero $b \in R$ annihilated by $R \cap I$ and thus by uI. Then I is annihilated by ub; localizing at a prime P containing the annihilator of ub, we find that $I_P \ncong R_P$, so I is not invertible.

Next let $I, J \subset K(R)$ be invertible. We first show that the natural surjection $I \otimes J \to IJ$ is injective. It suffices to show that for any prime P of R the map $I_P \otimes_{R_P} J_P \to (IJ)_P \subset K(R)_P$ is injective. Now $K(R_P)$ is a localization of $K(R)_P$ and it suffices to show that the composite map to $K(R_P)$ is injective. Thus we may assume from the outset that R is local. In this case $I \cong J \cong R$, so I and J are generated as R-modules by non-zero-divisors $s, t \in K(R)$, whence st is a non-zero-divisor. The composite map $R \cong R \otimes R \cong I \otimes J \to IJ = Rst \subset K(R)$ is then multiplication by st and so indeed injective. Next we show that the natural map $I^{-1}J \to \hom_R(I, J)$ sending t to ϕ_t is an isomorphism. By the previous part we can find a non-zero-divisor $v \in R \cap I$. If t is a nonzero element of $I^{-1}J$, then $tv \neq 0$, so t induces a nonzero element of $\hom_R(I, J)$ and the map $I^{-1}J \to \hom_R(I, J)$ is one-to-one. To show that it is onto, let $\phi \in \hom_R(I, J)$ be arbitrary and set $\phi(v) = w$. We claim that $\phi = \phi_{w/v}$; in fact, we claim that if any two homomorphisms ϕ, ψ from I to $K(R)$ agree on v, then they coincide; it suffices to show that this true after localization. The element v corresponds to a non-zero-divisor in R_P under the isomorphism $I_P \cong R_P$. The localizations ϕ_P, ψ_P may then be regarded as homomorphisms from R_P to $K(R)_P$ agreeing on the non-zero-divisor v. But then they must agree on 1, so everywhere.

Finally, let $I \subset K(R)$ be an invertible module. By the previous part the isomorphism $I^* \otimes I \to R$ may be identified with the multiplication map $I^{-1} \otimes I \to R$, so $I^{-1}I = R$. Conversely, if $I \subset K(R)$ is an R-submodule with $I^{-1}I = R$, then we can localize, supposing that R is local with maximal ideal P; we must show that $I \cong R$. By hypothesis there is $v \in I^{-1}$ with $vI \not\subset P$, forcing $vI = R$. But then v is not a zero divisor, so multiplication by v is an isomorphism from I to R.

Since the tensor product is associative, the set of isomorphism classes of invertible

R-modules forms a group under the tensor product, with the identity being the class of R and inverse of the class of I being that of I^*. This group is called the *Picard group* of R and is denoted $\mathrm{Pic}(R)$. Similarly, the set of invertible submodules of $K(R)$ is a group under multiplication, with the inverse of I being I^{-1}. This is called the group of *Cartier divisors* of R and is denoted $C(R)$. Thanks to the second part of the last theorem, the natural map $C(R) \to \mathrm{Pic}(R)$ is surjective; it takes the *principal divisor* generated by any unit of $K(R)$ to the identity. We have $Ru = Rv$ if and only if u, v differ by a unit in R, so the group of principal divisors under multiplication is identified with the quotient group $K(R)^*/R^*$. If I is an invertible divisor (= invertible module) and Ru a principal visiro, then $(Ru)I = uI$. We now claim that that if $I, J \subset K(R)$ are invertible divisors and if $\phi : I \to J$ is an isomorphism, then $I = uJ$ for some $u \in K(R)^*$. Since $\hom_R(I, J) = I^{-1}J$, any such ϕ is certainly multiplication by some $u \in K(R)$; similarly its inverse is multiplication by some $v \in K(R)$. Since I contains a non-zero-divisor, the product uv must equal 1, so u, v are indeed units in $K(R)$. Hence *the kernel of the natural map $C(R) \to Pic(R)$ is isomorphic to $K(R)^*/R^*$*. Moreover, *the group $C(R)$ is generated by the invertible ideals of R*, for if $I \subset K(R)$ is invertible, then I^{-1} contains a non-zero-divisor $a \in R$, forcing $aI \subset R, I = aI(a)^{-1}$.