LECTURE 5-1

Continuing with valuation rings (Chapter 5 of Atiyah-Macdonald) we now relate general valuation rings more tightly to the discrete ones we studied earlier. Let G be a totally ordered additive abelian group, so that there is a subset P of G such that $0 \notin P$, for every $g \in G$ with $g \neq 0$ we have that exactly one of $g, -g \in P$, and finally $g + h \in P$ whenever $g, h \in P$. Define the total order $\leq$ on G via $g \leq h$ if $h = g$ or $h - g \in P$; one checks immediately that this relation is reflexive, antisymmetric, and transitive. Given a field K, a *G-valuation* on it, or just a valuation, is a map $v : K^* \to G$ such that $v(xy) = v(x) + v(y)$ for all x, y and $v(x + y) \geq \min(v(x), v(y)$ whenever $x + y \neq 0$. We call G the *value group*. The corresponding valuation ring A of K is the subring consisting of all elements x with $v(x) \geq 0$, together with 0; conversely, if A is a valuation ring of K, U the group of units in A, and v the canonical map from K^* to the (multiplicative) group $G = K^*/U$, then we can totally order G by decreeing that its positive elements are exactly the cosets kU with $k \in A$ and v becomes a G-valuation with valuation ring A. The ideal structure of A is not quite as nice in general as in the discrete case, but is still fairly simple: given any $x, y \in A$ with $v(x) \leq v(y)$, we have $yx^{-1} \in A$, so the ideal (x) contains (y). It follows easily that a nonzero ideal I of A is completely determined by the set $v(I)$ of values of its nonzero elements, and in fact consists of all elements of A whose value lies in $v(I)$, together with 0. In turn $v(I)$ is an *upper ideal* of G, that is a subset of G containing $y \in G$ whenever it contains $x \in G$ and $x \leq y$; conversely any upper ideal of G is $v(I)$ for a unique ideal I . Thus, while A need not be a PID, *every finitely generated ideal of A is principal*; also, given two ideals I, J of A we have either $I \subset J$ or $J \subset I$. In homework for next week you will characterize the prime ideals of A and will also show that A is Noetherian if and only if it is a DVR (so that $G \cong \mathbb{Z}$ as an ordered abelian group).

Moreover, it turns out that any totally ordered abelian group G is the value group of a valuation ring. To see this we return to an old friend from the fall quarter, in a somewhat more general setting, namely the group algebra $\mathbb{Z}G$, consisting by definition of all finite integral combinations of elements of G (regarded as a multiplicative group), where multiplication is defined via multiplication in G and the distributive law. Then $\mathbb{Z}G$ is an integral domain, for given a nonzero element $\sum x_i g_i$ in it with $x_i \in \mathbb{Z}, g_i \in G$, then we may assume that the x_i are nonzero and $g_1 < \cdots < g_n$; call $x_1 g_1$ the lowest degree term of x. Then the lowest degree term of a product xy is the product of the lowest degree terms of x and y, so no product of nonzero elements can be 0. Thus $\mathbb{Z}G$ has a quotient field K. Define a valuation v on $\mathbb{Z}G$ by declaring the value of any sum $\sum_i x_i g_i$ as above to be $g_1 \in G$, where $x_1 g_1$ is its lowest degree term, and extend v to K^* by decreeing that $v(a/b) = v(a)v(b)^{-1}, a, b \in \mathbb{Z}G$; then it is easy to check that v is indeed a valuation on K with value group G. (The valuation ring of K is *not* $\mathbb{Z}G$, however, and need not even contain it; it consists of all fractions f with $v(f)$ positive in G). To wrap up valuations, we observe that *if A is a subring a field K, then the integral closure $\overline{A}$ of A in K is the intersection of all the valuation rings of K containing A*; indeed, since any valuation ring of K containing A is integrally closed, it must also contain $\bar{A}$; and conversely, if $x \notin \overline{A}$, then the ring $A' = A[x^{-1}]$ does not contain x, whence x^{-1} is a non-unit in A' and lies in a maximal ideal M' of A'. Letting Ω be an algebraic closure of $K' = A'/M'$; then we get a homomorphism from A to Ω by restricting the natural one defined on A'. Extending the domain of this homomorphism to a valuation ring $B \supset A$, we see that $x \notin B$, since x^{-1} maps to 0.

We conclude our excursion back to Atiyah-Macdonald with a brief account of the sheaf of rings attached to the spectrum of any ring A. A basic open subset of Spec A is the set U_f of prime ideals not containing any power of $f \in A$; any open subset of Spec A is the union of such sets. To U_f we attach the localization $A(U_f) = A_{(f^n)}$ by all powers of f. If another basic open set U_g is contained in U_f, then every prime ideal excluding g also excludes all powers of f, whence the image of g must lie in the nilradical of $A/(f)$ and some power g^n of g lies in (f), so equals uf for some $u \in A$. We thus get a map from $A(U_f)$ to $A(U_g)$ in this situation sending a/f^m to au^m/g^{mn}. This map depends only on U_f and U_g and is called the *restriction homomorphism*. If $U_f = U_g$, then the restriction homomorphism is the identity map; if $U \supset U' \supset U$" are all basic open sets, then the restriction map from U to U" is the composition of the ones from U to U' and from U' to U". Now if $P \in$ Spec A, then it turns out that the localized ring A_P may be viewed as the direct limit of the rings $A(U_f)$ as U_f runs through the basic open sets containing P. We call the collection of rings $A(U_f)$ a *presheaf of rings*; A_P is the called the *stalk* of the presheaf at P. The presheaf actually turns out to be a *sheaf* of rings, thanks to the following gluing property: given an element s_i of each basic open set U_i such that the images of s_i and s_j are equal in $A(U_i \cap U_j)$, there is a unique $s \in A$ whose image in $A(U_i)$ is s_i for all i. A *scheme* is then a ringed space (a topological space equipped with a sheaf of rings (one ring attached to each open subset), such that every point has a neighborhood homeomorphic to the ringed space attached as above to the spectrum of some ring.