LECTURE 4-5

We now return to a topic from last quarter that arose again in this week's homework, namely attaching finitely many prime ideals in a Noetherian ring R to a finitely generated module M over it. This material comes from Chapter 3 of Eisenbud. Recall that last quarter we started by looking at the annihilator Ann m of all $m \neq 0$ in M and we chose such an m so that Ann m was maximal among such annihilators. Then $P_1 =$Ann m need not be maximal as an ideal of R, but it is always prime, for if $xy \in$ Ann m but $x \notin$ Ann m, then Ann xm contains and so must equal Ann m, whence $y \in$ Ann m and $ym = 0$, as required. Letting M_1 be the submodule of M generated by m and passing to the quotient M/M_1, we find a submodule M_2 of M containing M_1 such that $M_2/M_1 \cong R/P_2$ with P_2 another prime ideal of R; iterating this procedure, we arrive at a chain of submodules $M_1 \subset M_2 \subset \ldots$ of M, which must terminate after finitely many steps at $M_n = M$, since M is Noetherian, such that the ith graded piece M_i/M_{i-1} of the filtration M_i) of M takes the form R/P_i for some prime ideal P_i of R (here $M_0 = 0$). Neither this filtration nor the ideals P_i are unique, in general, but we saw last quarter that if P_i is minimal among $P_1, \ldots, P_n$, then the number of indices j with $P_j = P_i$ is uniquely determined by M, as the dimension of the vector space M_{P_i} over the field $(R/P_i)_{P_i}$. (A simple example showing that the nonminimal P_i depend on the filtration occurs if we set $M = R/P, P$ a prime ideal of R; then we could clearly take $M_0 = 0, M_1 = M$, but we could also let $x \in M, x \neq 0$, take M_1 to be the submodule generated by x, whose annihilator is still P by primeness, and then $M_1 \neq M$, so we would still have M_2 to define; note that the annihilator of any graded piece M_i/M_{i-1} with $i \geq 2$ would contain x and so properly contain P.) Call the minimal P_i among $P_1, \ldots, P_n$ the *isolated primes* of M and the others the *embedded primes*. The terminology comes from algebraic geometry, where we study prime ideals by studying their varieties; the variety of a larger ideal is embedded in the one for a smaller one (and so is in some sense invisible).

Call a prime ideal P of R an *associated prime* of M (and denote by Ass M the set of all such) if it is the annihilator of some element of M. We have just seen that this set is nonempty for every M. If $0 \to M' \to M \to M" \to 0$ is a short exact sequence, then Ass M contains Ass M' and lies in the union of this set and Ass $M"$; to see this, observe that if $P \in$ Ass M but $P \notin$ Ass M', then any nonzero element of the submodule N generated by an element m of M with annihilator P again has annihilator P, by primeness, whence the intersection $N \cap M' = 0$ and N identifies with a submodule of $M"$, forcing $P \in$Ass $M"$, as desired. Applying this last fact repeatedly to the filtration (M_i) above, we see that *Ass M is finite and in fact lies in the set $\{P_1, \ldots, P_n\}$ of prime ideals arising from it.* Now the embedded primes attached to M may or may not lie in Ass M, but the isolated primes P_i always do; to see this, we look at the associated primes of the localized module M_{P_i} which by definition of localization consist exactly of the (localizations of) the associated primes of M contained in P_i. By the above argument and minimality of P_i, the only candidate for such an ideal is P_i itself; but Ass M_{P_i} is nonempty, so P_i must occur as an associated prime of M. Note also that the isolated primes P_i are exactly the minimal primes over the annihilator Ann M of M itself. To prove this we first observe that the product $P_1 \cdots P_n$ of prime ideals annihilates M, whence any prime ideal doing the same must contain one of the isolated P_i; conversely, the annihilator of M lies in every isolated P_i, since any

x annihilating M must also annihilate the subquotient R/P_i of M. Hence the *support* Supp M, consisting by definition of all primes P such that the localization M_P is nonzero, coincides with the set of primes containing Ann M, or equivalently some isolated P_i.

Now it turns out that there is a very convenient way to focus attention on particular associated primes of a module M, corresponding to decomposing an affine variety into its irreducible constituents. Call a submodule N of M *P-primary* if P is the only associated prime of the quotient M/N. If this holds and a product $xy \in R$ lies in Ann M/N) but $x \notin$ Ann M/N, then by passing to an element $xm \in xM/N$ whose annihilator is maximal among annihilators of nonzero elements of xM/N, we see that such an annihilator must be P, whence $y^n \in P$ for some n. Thus an ideal I of ring R is P-primary as a submodule of R if and only if it is primary in the sense defined earlier and has radical P (but not every ideal with radical P is P-primary). Now, arguing as we have done several times before by Noetherian induction, we see that every submodule of M is a finite intersection of submodules that are *irreducible* in the sense that none of them is the intersection of two submodules properly containing it (this has *nothing* to do with the usual notion of irreducible module and pertains only to submodules of a fixed one). Now every irreducible submodule N is primary. Otherwise, M/N would have at least two associated primes P, Q and accordingly two submodules, one isomorphic to R/P and the other to R/Q. Since the annihilator of every nonzero element of R/P is P and similarly for R/Q, these two submodules intersect in 0, whence N is the intersection of two properly larger submodules, a contradiction. We will explore primary decomposition further in subsequent lectures.